Unbiased Covariance Estimation with Interpolated Data

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Introduction

■ Estimation of covariance between financial assets (co-volatility, cross-volatility)
$\Rightarrow$ Portfolio risk, etc.

■ High-frequency data (Intraday data)
e.g. Hourly data, 30 minutes data, $\cdots$, 5 minutes data
$\Rightarrow$ Realized Volatility, **Realized Covariance**

■ However, RC has a serious problem:

Bias toward 0 (Epps effect)

Epps effects

Correlation of Log Returns for Three Stocks
(Epps, 1979, JASA)

Frequency	Chrysler-Ford	Chrysler-GM	Ford-GM
10 minutes	-0.014	0.007	0.055
20 minutes	0.017	0.026	0.118
40 minutes	0.041	0.040	0.197
1 hour	0.023	0.065	0.294
2 hours	0.112	0.129	0.383
3 hours	0.361	0.518	0.519

‘The higher frequency, the less correlation’

- Data Generating Process:

$$\underbrace{dp(t)}_{n\times 1} = \underbrace{\Sigma(t)}_{n\times n}\underbrace{dz(t)}_{n\times 1}, \quad 0 \le t \le T$$

where $z(t)$ is standard Brownian motion.

- (Instantaneous or spot) volatility matrix:

$$\Omega(t) \equiv \Sigma(t)\,\Sigma(t)'$$

$\omega_{ii}(t)$: volatility, $\omega_{ij}(t)$: covolatility.

Estimation of integrated volatility $\int_0^T \Omega(t)dt$

$\Rightarrow$ Quadratic variation:

$$p - \lim_{M\to\infty} \underbrace{\sum_{m=1}^{M} \Delta p(mT/M)\Delta p(mT/M)'}_{\text{Realized volatility matrix}} = \int_0^T \Omega(t)dt$$

Realized volatility matrix $\begin{cases} \text{Realized volatility} & \text{diagonal} \\ \text{Realized covariance} & \text{off-diagonal} \end{cases}$

Intuitively,

$$\int_0^T dp(t)dp(t)' = \int_0^T \Sigma(t)\underbrace{dz(t)dz(t)'}_{dtI_n}\Sigma(t)'$$

■ If all assets are observed at the same time points $\{mT/M\}_{m=0}^{M}$ (synchronous and equidistant sampling), there is no problem...

Nonsynchronous trading (observation)

■ Financial assets are traded (observed) at different time points.

1st asset: $0 = t_{0_1} < t_{1_1} < \cdots < t_{n_1} < \cdots < t_{N_1-1} < t_{N_1} = T$

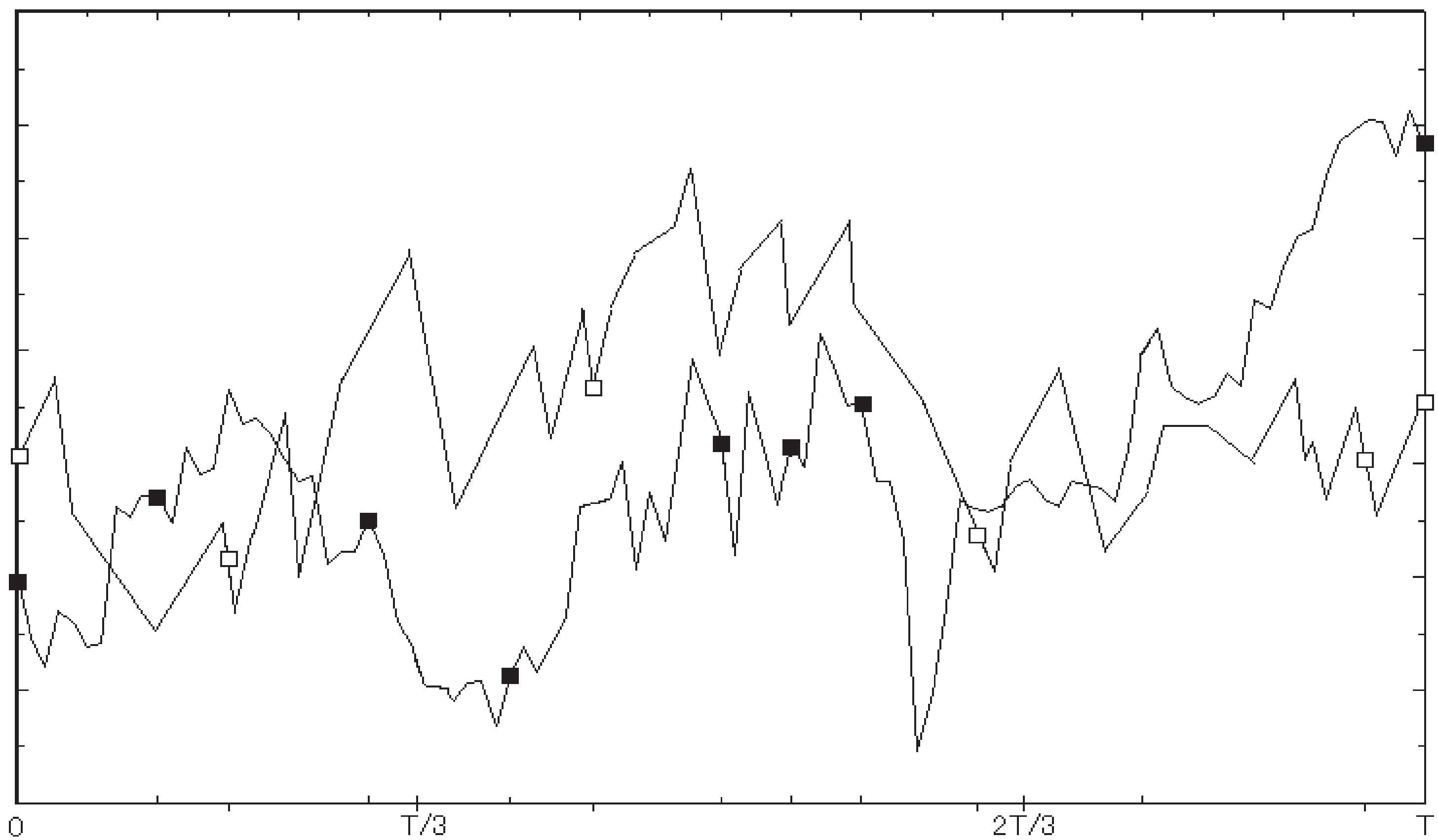
2nd asset: $0 = t_{0_2} < t_{1_2} < \cdots < t_{n_2} < \cdots < t_{N_2-1} < t_{N_2} = T$

Transaction data:

$$\overbrace{p_1(t_{0_1}), p_1(t_{1_1}), \cdots, p_1(t_{n_1}), \cdots, p_1(t_{N_1-1}), p_1(t_{N_1})}^{N_1+1 \text{ observations}}$$

$$\overbrace{p_2(t_{0_2}), p_2(t_{1_2}), \cdots, p_2(t_{n_2}), \cdots, p_2(t_{N_2-1}), p_2(t_{N_2})}^{N_2+1 \text{ observations}}$$

Nonsynchronous trading



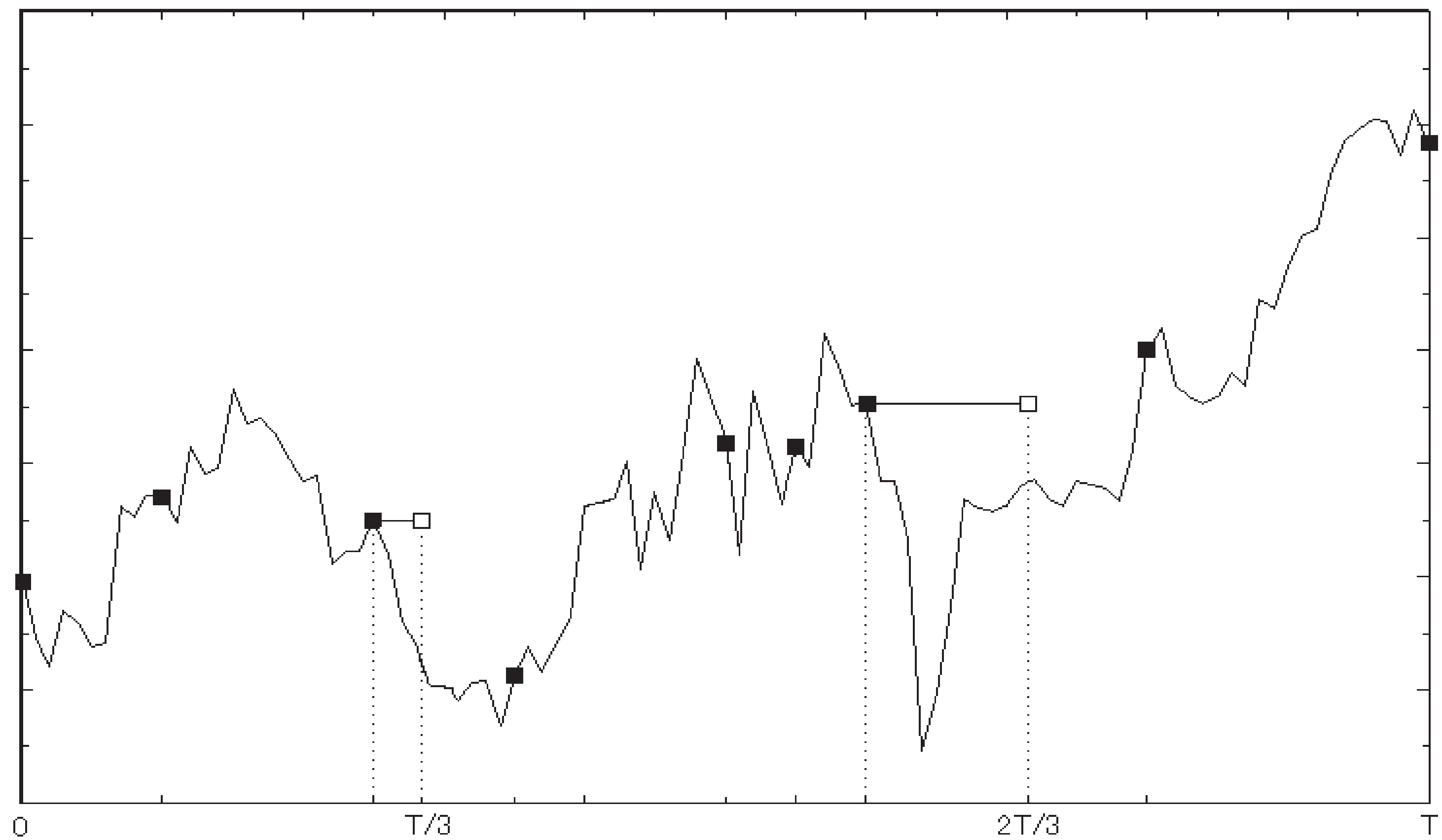
■ 1st asset ($N_1 = 7$); □ 2nd asset ($N_2 = 5$)

- $$\overbrace{\{p_i(t_{n_i})\}_{n_i=0}^{N_i}}^{\text{Unevenly spaced data}} \longrightarrow \boxed{\text{Interpolation}} \longrightarrow \overbrace{\{q_i(mT/M)\}_{m=0}^{M}}^{\text{Evenly spaced data}}$$

Previous-tick interpolation

$$q_i\left(\frac{mT}{M}\right) = p_i\left(\max\left(t_{n_i} : t_{n_i} \leq \frac{mT}{M}\right)\right)$$

Previous-tick interpolation ($M = 3$)



■ Raw data; □ Interpolated data

Realized covariance from the interpolated data

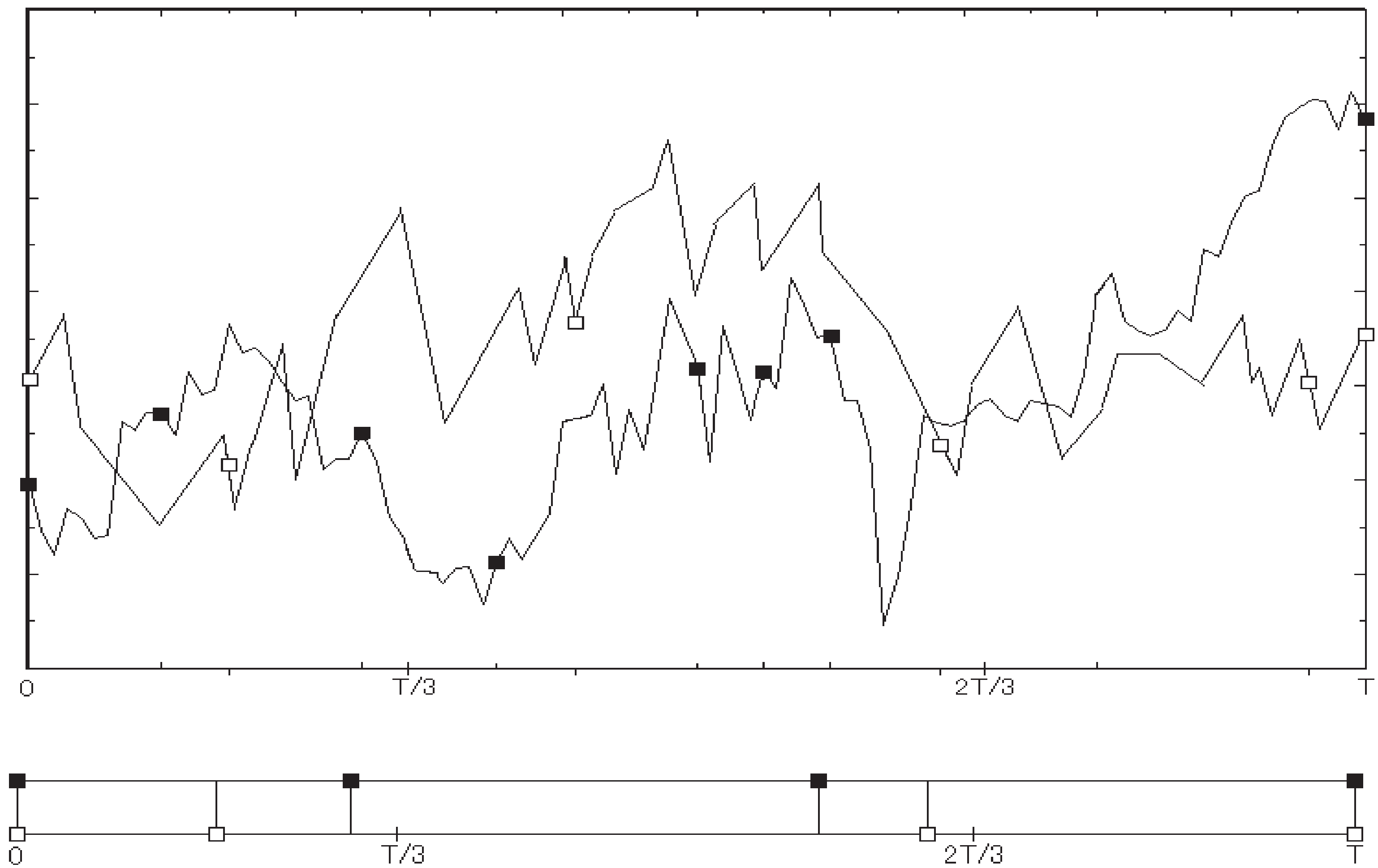
■ Realized covariance (An estimator of $\int_0^T \omega_{ij}(t)dt$)

$$RC(M) =$$
$$\sum_{m=1}^{M} \left\{ q_i\left(\frac{mT}{M}\right) - q_i\left(\frac{(m-1)T}{M}\right) \right\} \left\{ q_j\left(\frac{mT}{M}\right) - q_j\left(\frac{(m-1)T}{M}\right) \right\}$$

$\Rightarrow$ The bias toward 0

$$E(RC(M)) \to 0 \text{ as } M \to \infty$$

$\Rightarrow$ Epps effect (Epps, 1979)



Time positions of previous ticks

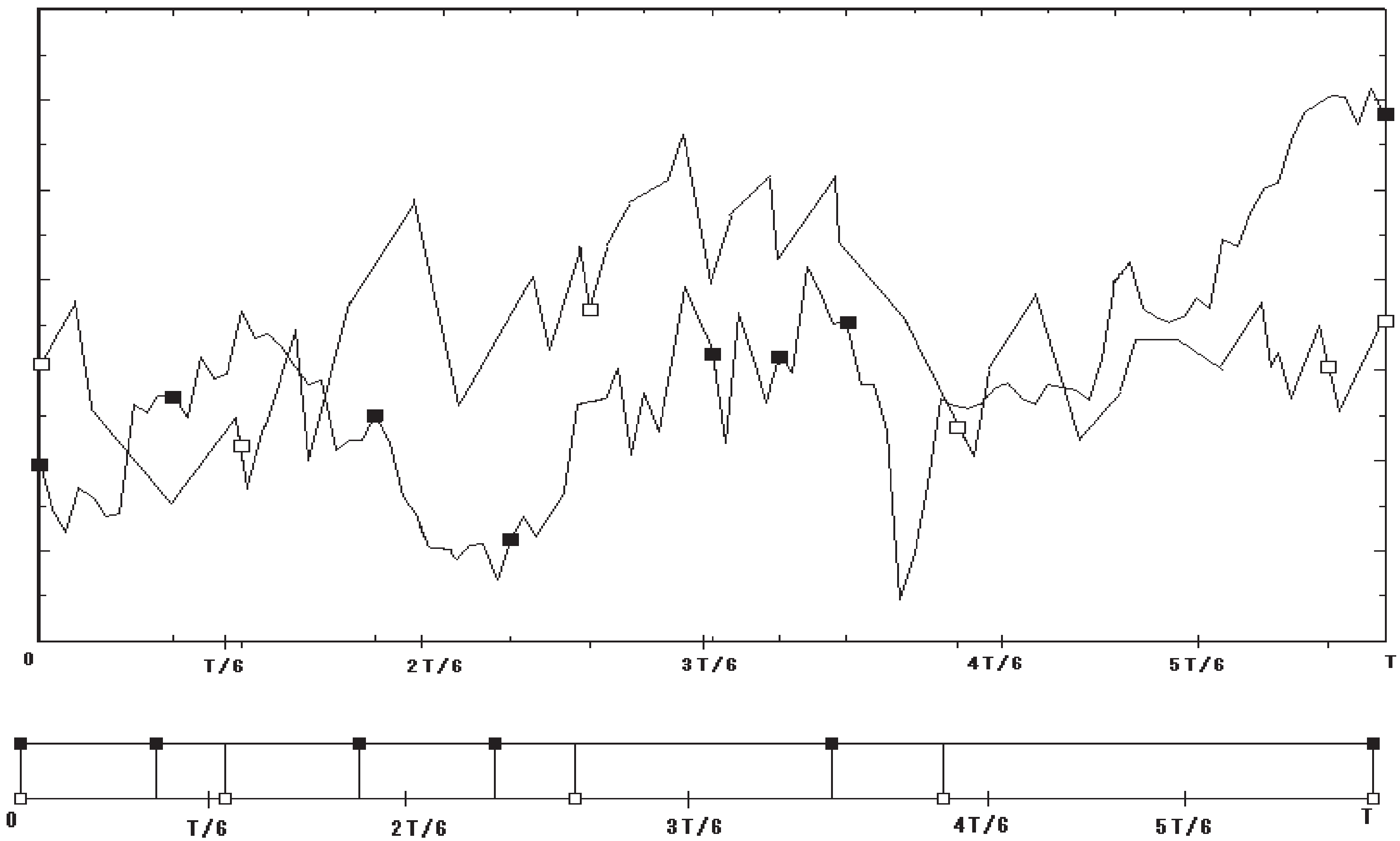
For simple notation,

$$\boldsymbol{q_i(m) \equiv q_i(mT/M), \Delta q_i(m) \equiv q_i(mT/M) - q_i((m-1)T/M)}$$

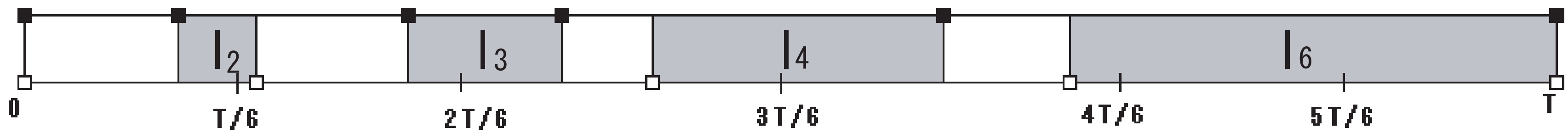
■ The expectation of realized covariance:

$$\begin{aligned} E(RC(3)) &= E\{\Delta q_1(1)\Delta q_2(1) + \Delta q_1(2)\Delta q_2(2) + \Delta q_1(3)\Delta q_2(3)\} \\ &= \int_{I_1} \omega_{12}(t)dt + \int_{I_2} \omega_{12}(t)dt + \int_{I_3} \omega_{12}(t)dt \end{aligned}$$

More serious case ($M = 6$)



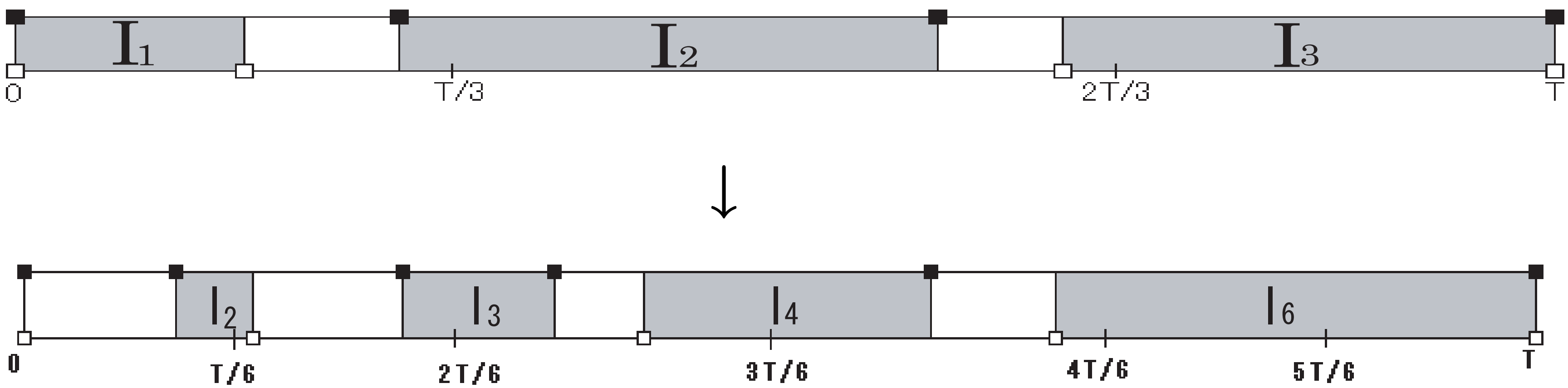
$$M = 6$$



$$q_1(4) = q_1(5),\ \ q_2(0) = q_2(1),\ \ q_2(4) = q_2(5)$$
$$\Rightarrow \quad \Delta q_1(5) = \Delta q_2(1) = \Delta q_2(5) = 0$$

$$\begin{aligned} E(RC(6)) &= E\{\Delta q_1(1) \overbrace{\Delta q_2(1)}^{0} + \Delta q_1(2)\Delta q_2(2) + \Delta q_1(3)\Delta q_2(3) \\ &\quad + \Delta q_1(4)\Delta q_2(4) + \underbrace{\Delta q_1(5)}_{0} \underbrace{\Delta q_2(5)}_{0} + \Delta q_1(6)\Delta q_2(6)\} \\ &= \int_{I_2} \omega_{12}(t)dt + \int_{I_3} \omega_{12}(t)dt + \int_{I_4} \omega_{12}(t)dt + \int_{I_6} \omega_{12}(t)dt \end{aligned}$$

$$M = 3$$



$$M = 6$$

‘The higher frequency, the smaller covered interval’

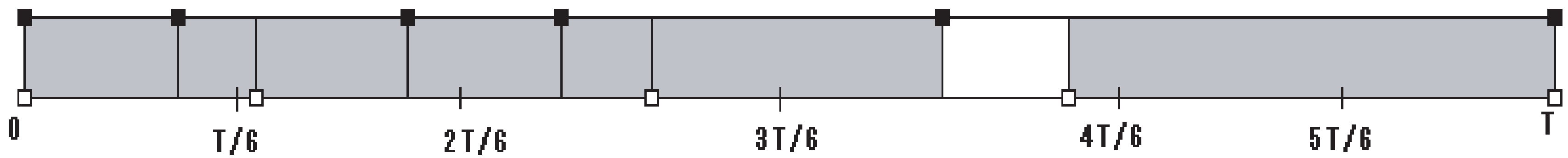
Lead & Lag modification

Bias-corrected estimator = Realized Covariance + Lead & Lag Terms

$$
\begin{aligned}
& RC(3) + \overbrace{\sum_{m=2}^{3} \Delta q_1(m-1)\Delta q_2(m) + \sum_{m=2}^{3} \Delta q_1(m)\Delta q_2(m-1)}^{\text{lead-lag terms}} \\
= \; & RC(3) + \Delta q_1(1)\Delta q_2(2) + \Delta q_1(2)\Delta q_2(1) \\
& \qquad + \Delta q_1(2)\Delta q_2(3) + \Delta q_1(3)\Delta q_2(2)
\end{aligned}
$$

Simple lead-lag modification is not enough...$\Rightarrow$

the area covered by RC + Lead-Lag terms



We need another step before lead-lag modification⇒

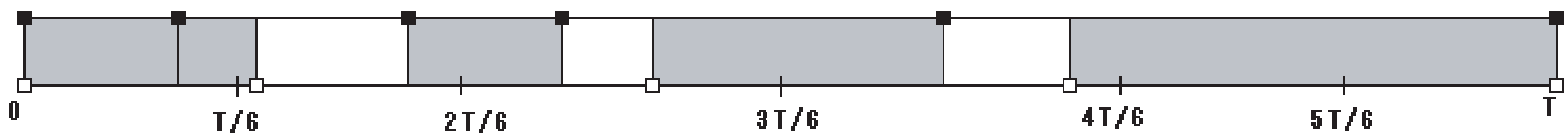
First step (Modification for zero returns)

Price change vectors

$$
\begin{array}{lccccccc}
(& \Delta q_1(1), & \Delta q_1(2), & \Delta q_1(3), & \Delta q_1(4), & \Delta^2 q_1(6) &) \\
(& \multicolumn{2}{c}{\Delta^2 q_2(2),} & \Delta q_2(3), & \Delta q_2(4), & \Delta^2 q_2(6) &)
\end{array}
$$

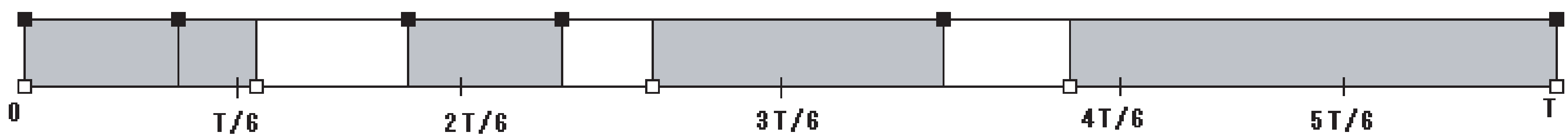
Contemporary part (sum up time-overlapping returns):

$$
\begin{aligned}
\Delta q_1(1)\Delta^2 q_2(2) + \Delta q_1(2)\Delta^2 q_2(2) + \Delta q_1(3)\Delta q_2(3) \\
+ \Delta q_1(4)\Delta q_2(4) + \Delta^2 q_1(6)\Delta^2 q_2(6).
\end{aligned}
$$



Second step (lead-lag modification)

$$
\begin{array}{llllll}
(& \Delta q_1(1), & \Delta q_1(2), & \Delta q_1(3), & \Delta q_1(4), & \Delta^2 q_1(6) \quad) \\
(& \multicolumn{2}{c}{\Delta^2 q_2(2),} & \Delta q_2(3), & \Delta q_2(4), & \Delta^2 q_2(6) \quad)
\end{array}
$$



Lead-Lag part:

$$
\begin{array}{lll}
\Delta q_1(2)\Delta q_2(3) & \Delta q_1(3)\Delta q_2(4) & \Delta q_1(4)\Delta^2 q_2(6) \\
\Delta^2 q_2(2)\Delta q_1(3) & \Delta q_2(3)\Delta q_1(4) & \Delta q_2(4)\Delta^2 q_1(6)
\end{array}
$$

Bias-corrected estimator:

$$
BC(6) = \text{Contemporary part} + \text{Lead-Lag part}
$$

Bias-corrected estimator

Similarly, in general case, the modification consists of two steps:
(1) zero-return modification (consider price change vectors and sum up products of time-overlapping changes)
(2) lead-lag modification (add products of lead-lag changes)

$$BC(M) = \sum_{m_1, m_2} 1_{A_1}(q_1(m_1) - q_1(m_1^-))1_{A_2}(q_2(m_2) - q_2(m_2^-))1_A$$

where

$$m_i^- = \max\{m_i' < m_i : q_i(m_i') \neq q_i(m_i' - 1)\},$$
$$A_i = \{q_i(m_i) \neq q_i(m_i - 1)\}$$
$$A = \left\{\left[m_1^-, m_1\right] \cap \left[m_2^-, m_2\right] \neq \emptyset\right\}$$

$RC(M)$ and $BC(M)$ for a large M

■ Hayashi and Yoshida (2005, Bernoulli)'s unbiased estimator for **"transaction data"**:

$$HY = \sum_{n_1,n_2} (p_1(t_{n_1}) - p_1(t_{n_1-1}))(p_2(t_{n_2}) - p_2(t_{n_2-1}))1_B$$

where $B = \{(t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset\}$.

■ For large M

$$RC(M) = 0, \tag{1}$$
$$BC(M) = HY. \tag{2}$$

Monte Carlo study

- DGP:

$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dz_1(t) \\ dz_2(t) \end{pmatrix}, \; 0 \le t \le T$$
$$d\sigma_{ij}(t) = \kappa \left(\theta - \sigma_{ij}(t)\right) dt + \gamma dz_{ij}(t), \; i,j = 1,2.$$

where $\kappa = 0.01$, $\theta = 0.01$, and $\gamma = 0.001$ and $T = 60 \times 60 \times 4.5$ seconds. (p_1 and p_2 are **positively** correlated in average)

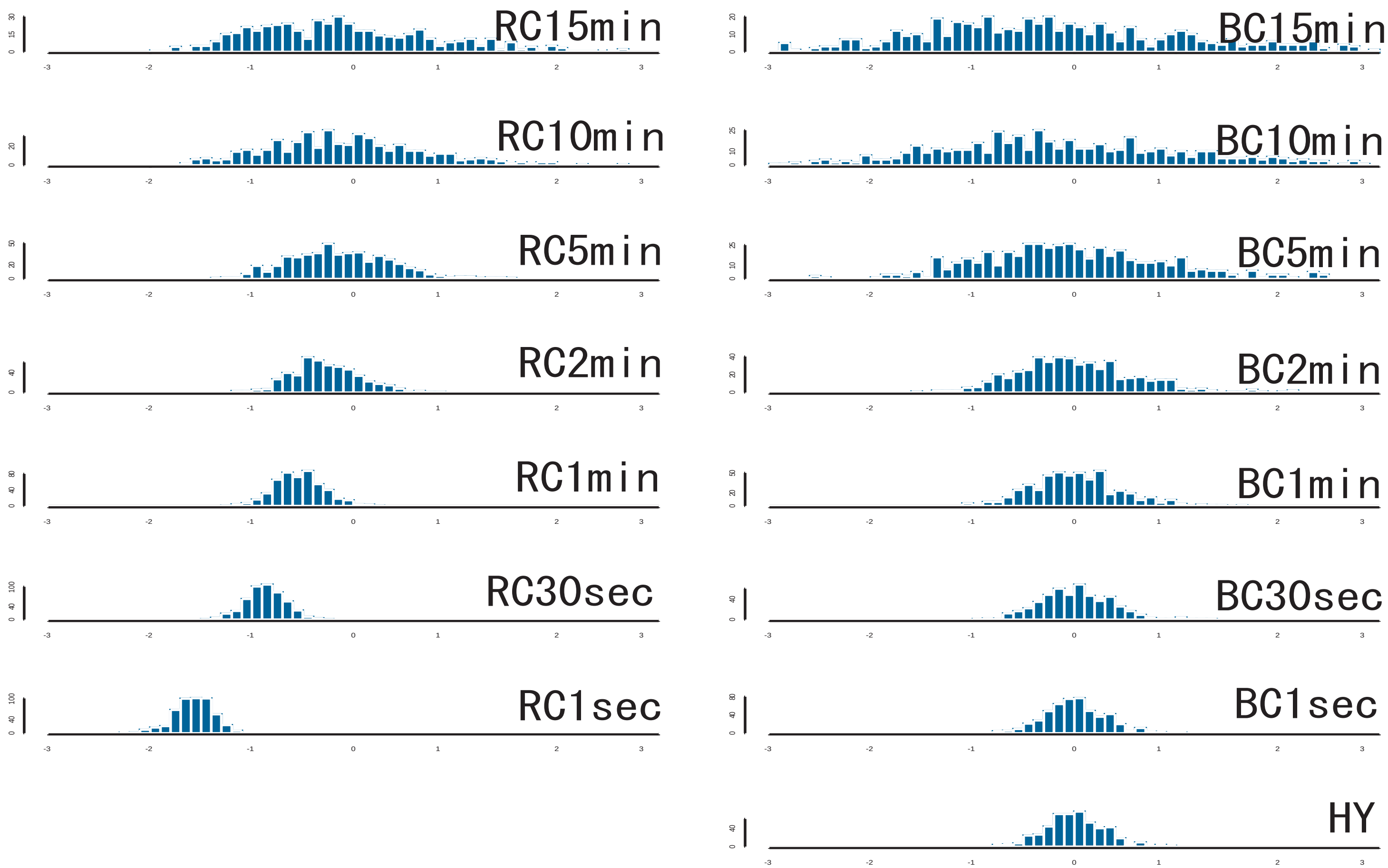
- Time differences are drawn from an exponential distribution:

$$F\left(t_{n_i} - t_{n_i-1}\right) = 1 - \exp\left\{-\lambda_i \left(t_{n_i} - t_{n_i-1}\right)\right\}, \; i = 1,2$$

where $F(\cdot)$ denotes a cumulative distribution function, $\lambda_1 = 23.4$ and $\lambda_2 = 20.9$.

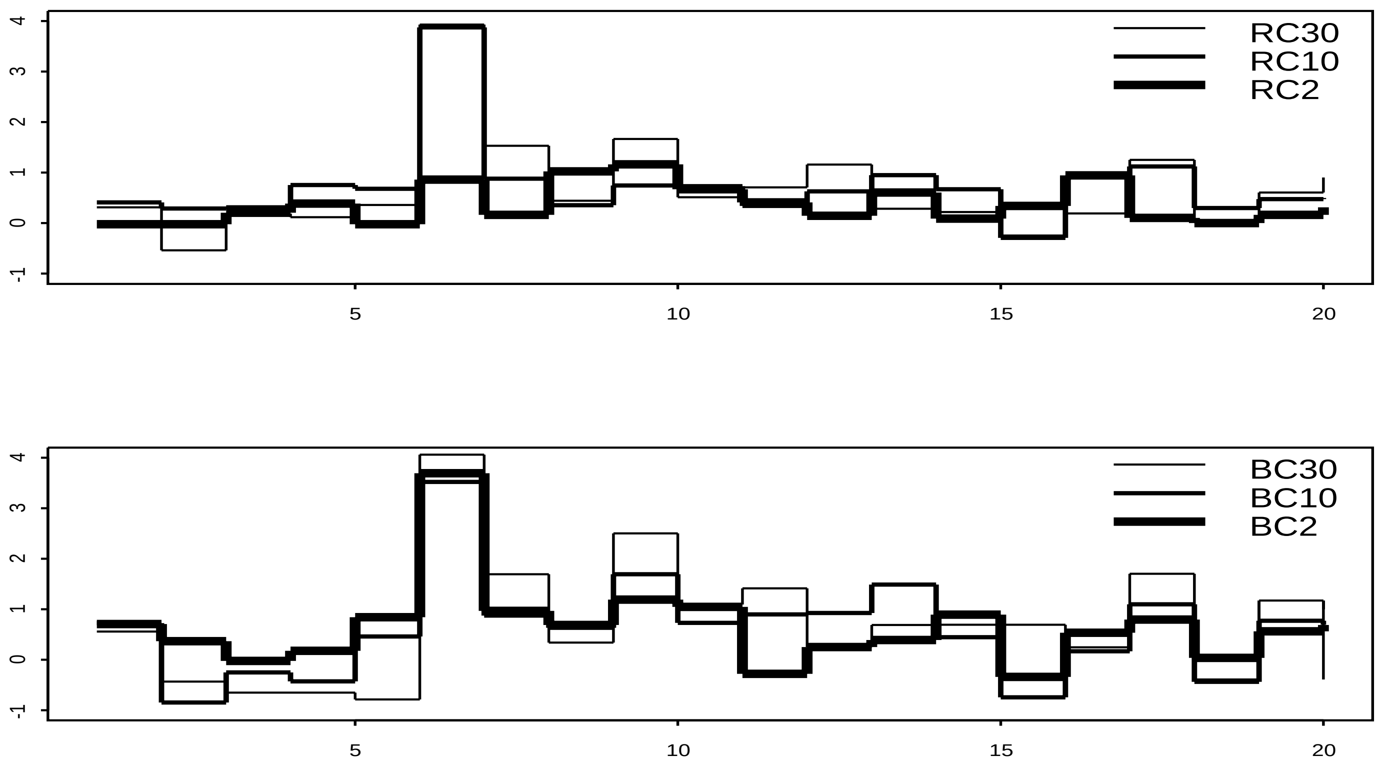
- The number of replications is 500.

Monte Carlo result



Distribution of measurement error (500 replications)

An empirical example



Daily RC and BC (20 days, Honda-Nissan)

Summary

- An explanation of the cause of the bias on RC
- A bias-corrected estimator

Remaining works:

- Microstructure noise (Observation error)
- Evaluation of Variance of the estimator

$\Rightarrow$ MSE analysis