Finite Sample Analysis of Weighted Realized Covariance with Noisy Asynchronous Observations

Taro Kanatani

JSPS Fellow, Institute of Economic Research, Kyoto University

October 18, 2007, Stanford

Introduction

■ Estimation of covariance between financial assets (co-volatility, cross-volatility)
⇒ Portfolio risk, etc.

■ High-frequency data (Intraday data)
e.g. Hourly data, $\cdots$, 1 minute data, $\cdots$, **transaction data**
⇒ Realized Volatility, **Realized Covariance**

■ Serious problems for using RC with too high frequency data (transaction data):
1. Nonsynchronous observation
2. **Microstructure noise (observation error)**
⇒ Weighted Realized Covariance (WRC)

No noise and synchronous observation

■ True logarithmic price process:

$$\underbrace{dp\,(t)}_{2\times 1} = \underbrace{\Sigma\,(t)}_{2\times 2}\underbrace{dz\,(t)}_{2\times 1}, \quad 0\ \le t \le T$$

where $z(t)$ is standard Brownian motion.

■ (Instantaneous or spot) volatility matrix:

$$\Omega(t) \equiv \Sigma\,(t)\,\Sigma\,(t)'$$

element by element (12 or 21 element)

$$\omega_{12}(t) = \sum_i \sigma_{1i}(t)\sigma_{2i}(t)$$

■ Discrete observation time points:

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_{N-1} < t_N = T$$

Realized covariance

Estimation of integrated covariance $\int_0^T \omega_{12}(t)dt$

e.g. $T = 1$ day, Daily Covariance

$$\text{plim}_{N\to\infty} \underbrace{\sum_{n=1}^{N} \Delta p_1(t_n)\Delta p_2(t_n)}_{\text{Realized covariance}} = \int_0^T \omega_{12}(t)dt$$

Intuitively,

$$\int_0^T dp_1(t)dp_2(t) = \int_0^T \sum_i \sigma_{1i}(t)\sigma_{2i}(t) \underbrace{dz_i(t)^2}_{dt}$$

■ If true prices are observed synchronously, there is no problem...

Nonsynchronous and noisy observation

■ Nonsynchronous observation: financial assets are traded (observed) at different time points.

$$\text{1st asset: } \mathbf{0} = t_{0_1} < t_{1_1} < \cdots < t_{n_1} < \cdots < t_{N_1-1} < t_{N_1} = T$$
$$\text{2nd asset: } \mathbf{0} = t_{0_2} < t_{1_2} < \cdots < t_{n_2} < \cdots < t_{N_2-1} < t_{N_2} = T$$

⇒ Nonsynchronous bias ⇒ Hayashi-Yoshida (2005) estimator :)

■ Market micirostructure noise: the price is observed with noise at each point.

$$\overbrace{p_i^o(t_{n_i})}^{\text{observed price}} = \overbrace{p_i(t_{n_i})}^{\text{true price}} + \overbrace{e_i(t_{n_i})}^{\text{noise(observation error)}}$$

where $e_i(t_{n_i}) \sim IID(0, \sigma_i^2)$

Effect of the noise in short interval

Define $r_i^o(t_{n_i}) = \Delta p_i^o(t_{n_i})$, $r_i(t_{n_i}) = \Delta p_i(t_{n_i})$, $u_i(t_{n_i}) = \Delta e_i(t_{n_i})$

$$r_i^o(t_{n_i}) = \underbrace{r_i(t_{n_i})}_{V(r)=O_p(\Delta t)} + \underbrace{u_i(t_{n_i})}_{V(u)=O(1)}$$

■ In short interval, the noise term swamps the true return!

True return is hidden by the noise!!

How to mitigate the noise?

■ For volatility

1. Discard data (Use low frequency data) $\Rightarrow$ Optimal frequency: Bandi and Russell (2005) etc
2. Realized kernel: Barndorff-Nielsen et al. (2007) $\supset$ Two Scale Estimator (Zhang et al., 2005) $\approx$ Hansen and Lunde (2005)'s estimator
3. Fourier estimator: Mancino and Sanfelici (2006)

■ For cross-volatility

1. Optimal frequency: Griffin and Oomen (2006), Bandi and Russell (2006)
2. Realized kernel: ?
3. Fourier estimator: ?

Weighted realized covariance

$$
\begin{aligned}
WRC &= \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} r_1^o(t_{n_1}) r_2^o(t_{n_2}) w_{n_1 n_2} = \underbrace{(r_1^o)'}_{1\times N_1} \overbrace{W}^{N_1\times N_2} \underbrace{r_2^o}_{N_2\times 1} \\
&= r_1' W r_2 + r_1' W u_2 + u_1' W r_2 + u_1' W u_2,
\end{aligned}
$$

where

$$
\begin{aligned}
r_i^o &= (r_i^o(t_{1_i}), ..., r_i^o(t_{n_i}), ..., r_i^o(T))', \\
r_i &= (r_i(t_{1_i}), ..., r_i(t_{n_i}), ..., r_i(T))', \\
u_i &= (u_i(t_{1_i}), ..., u_i(t_{n_i}), ..., u_i(T))'.
\end{aligned}
$$

■ WRC nests

1. Lower frequency methods
2. Realized kernel: $r_2 = r_1$, $w_{11} = ... = w_{NN}$, $w_{12} = ... = w_{N-1N}$, $w_{21} = ... = w_{NN-1}$,... (Toeplitz matrix)
3. Fourier estimator

Bias of WRC

Denote

$$\begin{cases} w^{diag}_{n_1 n_2} & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ w^{off}_{n_1 n_2} & \text{otherwise.} \end{cases}, \qquad IC = \int_0^T \omega_{12} dt,$$

$$\begin{cases} IC_{n_1 n_2} = \int_{\max\{t_{n_1-1}, t_{n_2-1}\}}^{\min\{t_{n_1}, t_{n_2}\}} \omega_{12}(t) dt & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

■ Bias of WRC:

$$E[WRC - IC] = \sum_{n_1, n_2} (w^{diag}_{n_1 n_2} - 1) IC_{n_1 n_2}$$

■ If $w^{diag}_{n_1 n_2} = 1$ for all n_1, n_2, WRC is unbiased. ($w^{diag}_{n_1 n_2} = 1$ and $w^{off}_{n_1 n_2} = 0 \iff$ Hayashi and Yoshida (2005) estimator)

MSE of WRC

$$\begin{aligned} & E[WRC - IC]^2 \\ = & \underbrace{E[r_1' W r_2 - IC]^2}_{A} + \underbrace{E[r_1' W u_2]^2}_{B} + \underbrace{E[u_1' W r_2]^2}_{C} + \underbrace{E[u_1' W u_2]^2}_{D} \end{aligned}$$

where

$$A = \sum \left(w_{n_1 n_2}^{diag} IC_{n_1 n_2} \right)^2 + \sum w_{n_1 n_2}^2 IV_{n_1} IV_{n_2} + Bias^2$$

$$B = \sigma_2^2 \sum w_{n_1 n_2} (2 w_{n_1 n_2} - w_{n_1 n_2 - 1} - w_{n_1 n_2 + 1}) IV_{n_1} (IV_{n_i} \equiv \int_{t_{n_i - 1}}^{t_{n_i}} \omega_{ii} dt)$$

$$C = \sigma_1^2 \sum w_{n_1 n_2} (2 w_{n_1 n_2} - w_{n_1 - 1 n_2} - w_{n_1 + 1 n_2}) IV_{n_2}$$

$$\begin{aligned} D = \sigma_1^2 \sigma_2^2 \sum w_{n_1 n_2} \{ & 4 w_{n_1 n_2} + w_{n_1 - 1 n_2 - 1} + w_{n_1 - 1 n_2 + 1} + w_{n_1 + 1 n_2 - 1} \\ & + w_{n_1 + 1 n_2 + 1} - 2(w_{n_1 - 1 n_2} + w_{n_1 + 1 n_2} + w_{n_1 n_2 - 1} + w_{n_1 n_2 + 1}) \} \end{aligned}$$

For feasible evaluation of MSE: weight function

■ A specific form of weight function: Hayashi-Yoshida, Fourier estimator, Error function weight, Kernels

■ We limit our discussion within unbiased estimators ($w^{diag}_{n_1 n_2} = 1$). $\Rightarrow$ $Bias = 0$ and $\sum \left(w^{diag}_{n_1 n_2} IC_{n_1 n_2} \right)^2$ is unknown constant therefore we never need to evaluate it when minimizing MSE!

For feasible evaluation of MSE: Approximation and estimation

■ Assumption: "Volatility does not change so much over $[0, T]$" (Bandi and Russell, 2005) $\Rightarrow$

$$IV_{n_i} \approx \frac{IV_i \Delta t_{n_i}}{T}$$

where $IV_i = \int_0^T \omega_{ii}(t)dt$.

■ Estimates of σ_i^2, $IV_i \Rightarrow$ Bandi and Russell (2005), Barndorff-Nielsen et al. (2007) etc.

Minimization of feasible MSE

$$\theta^* = \arg\min_{\theta}(A' + B' + C' + D')$$

where

$$A' = T^{-2} IV_1 IV_2 \sum w_{n_1 n_2}^2 \Delta t_{n_1} \Delta t_{n_2}$$
$$B' = T^{-1} IV_1 \sigma_2^2 \sum w_{n_1 n_2}(2w_{n_1 n_2} - w_{n_1 n_2 - 1} - w_{n_1 n_2 + 1}) \Delta t_{n_1}$$
$$C' = T^{-1} IV_2 \sigma_1^2 \sum w_{n_1 n_2}(2w_{n_1 n_2} - w_{n_1 - 1 n_2} - w_{n_1 + 1 n_2}) \Delta t_{n_2}$$
$$D' = \sigma_1^2 \sigma_2^2 \sum w_{n_1 n_2} \{4w_{n_1 n_2} + w_{n_1 - 1 n_2 - 1} + w_{n_1 - 1 n_2 + 1} + w_{n_1 + 1 n_2 - 1} + w_{n_1 + 1 n_2 + 1} - 2(w_{n_1 - 1 n_2} + w_{n_1 + 1 n_2} + w_{n_1 n_2 - 1} + w_{n_1 n_2 + 1})\}$$
$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1 - 1}, t_{n_1}] \cap (t_{n_2 - 1}, t_{n_2}] \neq \emptyset \\ f(t_{n_1}, t_{n_2}; \theta) & \text{otherwise,} \end{cases}$$

An example of WRC: Fourier estimator

■ Fourier estimator (Malliavin and Mancino, 2002)

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } t_{n_1} = t_{n_2} \\ \dfrac{\sin\frac{(n+1)(t_{n_1}-t_{n_2})}{2}\cos\frac{n(t_{n_1}-t_{n_2})}{2}}{n\sin\frac{(t_{n_1}-t_{n_2})}{2}} & \text{otherwise,} \end{cases}$$

where n is the number of Fourier coefficients.

■ The (nonsynchronous) bias corrected version is

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ \dfrac{\sin\frac{(n+1)(t_{n_1}-t_{n_2})}{2}\cos\frac{n(t_{n_1}-t_{n_2})}{2}}{n\sin\frac{(t_{n_1}-t_{n_2})}{2}} & \text{otherwise.} \end{cases}$$

An example of WRC: Error function weight

- Error function weight

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ \exp\left(-\frac{(t_{n_1}-t_{n_2})^2}{h}\right) & \text{otherwise.} \end{cases}$$

where $h > 0$.

- Extreme cases

$$h \to 0, \quad WRC = HY$$
$$h \to \infty, \quad WRC = (r_1^o)' 1_{N_1} 1'_{N_2} r_2^o = (p_1^o(T) - p_1^o(0))(p_2^o(T) - p_2^o(0))$$

An example of WRC: kernels in BHLS(2007)

- unbiased kernel

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset, \\ k\left(\frac{|t_{n_1} - t_{n_2}|}{H}\right) & \text{if } |t_{n_1} - t_{n_2}| < H, \\ 0 & \text{otherwise.} \end{cases}$$

	$k(x)$
Bartlett	$1 - x$
Epanechnikov	$1 - x^2$
Parzen	$1 - 6x^2 + 6x^3 \ (0 \le x \le 1/2)$
	$2(1 - x)^3 \ (1/2 < x \le 1)$
Tukey-Hanning	$(1 + \cos(\pi x))/2$
Mod. Tukey-Hanning	$(1 - \cos \pi (1 - x)^2)/2$
Tukey-Hanning p	$\sin^2(\pi(1 - x)^p/2)$

An example of WRC: Hayashi-Yoshida estimator

- Hayashi-Yoshida (2005) estimator:

$$HY = \sum_{n_1,n_2} r_1^o(t_{n_1}) r_2^o(t_{n_2}) I\{(t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset\},$$

where $I\{\cdot\}$ is indicator function.

- Hayashi-Yoshida weight

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

■ Griffin and Oomen (2006) estimator (Lower frequency version of HY):

$$HY(k) = \sum_{kn_1, kn_2} \Delta^k p_1^o(t_{kn_1}) \Delta^k p_2^o(t_{kn_2}) I\{(t_{kn_1-k}, t_{kn_1}] \cap (t_{kn_2-k}, t_{kn_2}] \neq \emptyset\},$$

where k is a positive integer.

■ Weight matrix:

$$\overbrace{\begin{pmatrix} \sum_{n_1=1}^{k} r_i(t_{n_1}) \\ \sum_{n_1=k+1}^{2k} r_i(t_{n_1}) \\ \vdots \end{pmatrix}}^{r_1^o(k):[N_1/k]\times 1} = \overbrace{\begin{pmatrix} 1_k & 0_{N_1-k} & \\ 0_k & 1_k & 0_{N_1-2k} \\ \vdots & \vdots & \vdots \end{pmatrix}}^{A_1(k):[N_1/k]\times N_1} r_1$$

where $1_k = (1 \cdots 1), 0_k = (0 \cdots 0)$.

$$HY(k) = (r_1^o(k))' D(k) r_2^o(k) = (r_1^o)' \overbrace{A_1(k)' D(k) A_2(k)}^{W} r_2^o$$

An example of WRC: Subsampling estimator

$$\overbrace{\begin{pmatrix} \sum_{n_1=1}^{i} r_i(t_{n_1}) \\ \sum_{n_1=i+1}^{i+k} r_i(t_{n_1}) \\ \vdots \end{pmatrix}}^{r_1^o(k_1)^i} = \overbrace{\begin{pmatrix} 1_i & 0_{N_1-i} & \\ 0 & 1_k & 0_{N_1-k-i} \\ \vdots & \vdots & \vdots \end{pmatrix}}^{A_1^i(k_1)} r_1$$

- Subsampling HY:

$$HY^{ss}(k1,k2) = (k_1 k_2)^{-1} \sum_{i,j} (r_1^o(k_1)^i)' D(k_1,k_2)^{ij} r_2^o(k_2)^j$$

$$= (r_1^o)' \underbrace{\frac{\sum_{i,j} A_1^i(k_1)' D(k_1,k_2)^{ij} A_2^j(k_2)}{k_1 k_2}}_{W} r_2^o$$

- $D(k_1, k_2) \rightarrow W(k_1, k_2)$ can reduece the MSE?

Monte Carlo study

■ Efficient price process:

$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dz_1(t) \\ dz_2(t) \end{pmatrix}, \ 0 \le t \le T$$

$$d\sigma_{ij}(t) = \kappa\left(\theta - \sigma_{ij}(t)\right)dt + \gamma dz_{ij}(t), \ i,j = 1,2.$$

where $\kappa = 0.1$, $\theta = 1$, $\gamma = 0.1$, $T = 1$(day), $\Delta = 1/60 \times 60 \times 4.5$(one second precision for Japanese stock exchanges).

■ Time differences are drawn from an exponential distribution:

$$F\left(t_{n_i} - t_{n_i-1}\right) = 1 - \exp\left\{-\lambda_i\left(t_{n_i} - t_{n_i-1}\right)\right\}, \ i = 1,2$$

where $F(\cdot)$ denotes a cumulative distribution function, $\lambda_i = 1/60\Delta$. (Average duration is 60 seconds)

■ Independent noise: $e_1(t_{n_1}) \sim NID(0, 0.025)$, $e_2(t_{n_2}) \sim NID(0, 0.05)$

Monte Carlo result 1

$\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.025)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.05)}$

	Sample MSE	Ave. of optimal parameter
HY	0.845	
$LHY(k^*)$	0.469	$\bar{k}^* = 7.37$
$MFE(n^*)$	0.242	$\bar{n}^* = 12.4$
$EF(h^*)$	0.146	$\bar{h}^* = 0.0293$
$BAR(H^*)$	0.145	$\bar{H}^* = 0.0509$
$EPA(H^*)$	0.185	$\bar{H}^* = 0.0405$
$PAR(H^*)$	0.147	$\bar{H}^* = 0.0673$
$TH(H^*)$	0.153	$\bar{H}^* = 0.0506$
$MTH(H^*)$	0.144	$\bar{H}^* = 0.0810$

Sample MSE $= (1/500) \sum_{r=1}^{500} \left(estimate^{(r)} - IC^{(r)} \right)^2$

Monte Carlo result 2

$$\boldsymbol{\lambda_i \Delta = 1/60},\ \boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)},\ \boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$$

	Sample MSE	Ave. of optimal parameter
$\boldsymbol{HY}$	0.118	
$\boldsymbol{LHY(k^*)}$	0.118	$\boldsymbol{\bar{k}^* = 1}$
$\boldsymbol{MFE(n^*)}$	0.117	$\boldsymbol{\bar{n}^* = 25.3}$
$\boldsymbol{EF(h^*)}$	0.0911	$\boldsymbol{\bar{h}^* = 0.0130}$
$\boldsymbol{BAR(H^*)}$	0.0907	$\boldsymbol{\bar{H}^* = 0.0226}$
$\boldsymbol{EPA(H^*)}$	0.0978	$\boldsymbol{\bar{H}^* = 0.0166}$
$\boldsymbol{PAR(H^*)}$	0.0920	$\boldsymbol{\bar{H}^* = 0.0314}$
$\boldsymbol{TH(H^*)}$	0.0949	$\boldsymbol{\bar{H}^* = 0.0231}$
$\boldsymbol{MTH(H^*)}$	0.0924	$\boldsymbol{\bar{H}^* = 0.0377}$

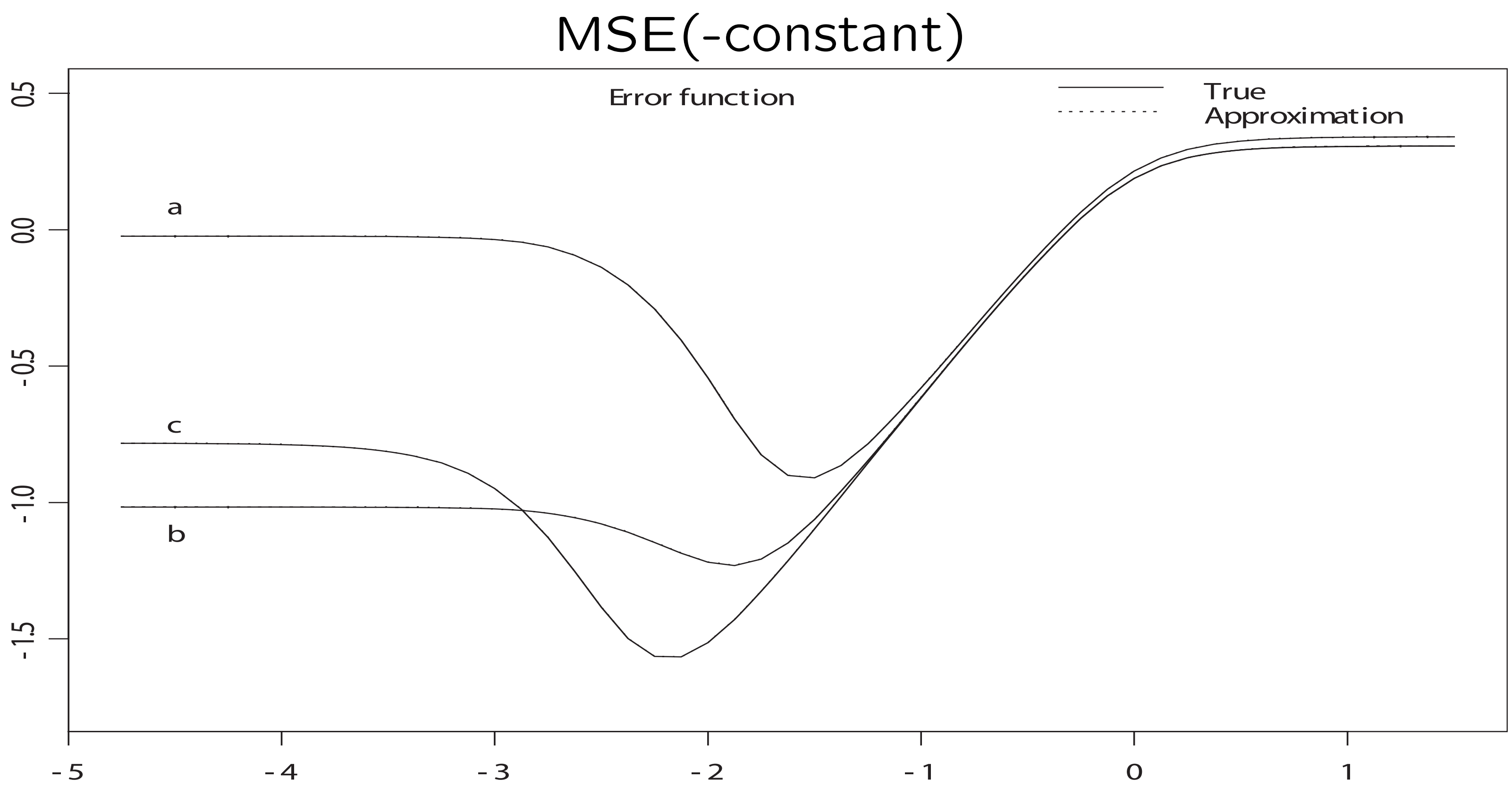
$$\text{Sample MSE} = \boldsymbol{(1/500) \sum_{r=1}^{500} \left(estimate^{(r)} - IC^{(r)} \right)^2}$$

Monte Carlo result 3

$\lambda_i \Delta = 1/15$, $e_1(t_{n_1}) \sim NID(0, 0.005)$, $e_2(t_{n_2}) \sim NID(0, 0.01)$

	Sample MSE	Ave. of optimal parameter
HY	0.168	
$LHY(k^*)$	0.105	$\bar{k}^* = 6.05$
$MFE(n^*)$	0.0485	$\bar{n}^* = 49.5$
$EF(h^*)$	0.0358	$\bar{h}^* = 0.00651$
$BAR(H^*)$	0.0348	$\bar{H}^* = 0.0117$
$EPA(H^*)$	0.0439	$\bar{H}^* = 0.00918$
$PAR(H^*)$	0.0368	$\bar{H}^* = 0.0158$
$TH(H^*)$	0.0380	$\bar{H}^* = 0.0117$
$MTH(H^*)$	0.0361	$\bar{H}^* = 0.0194$

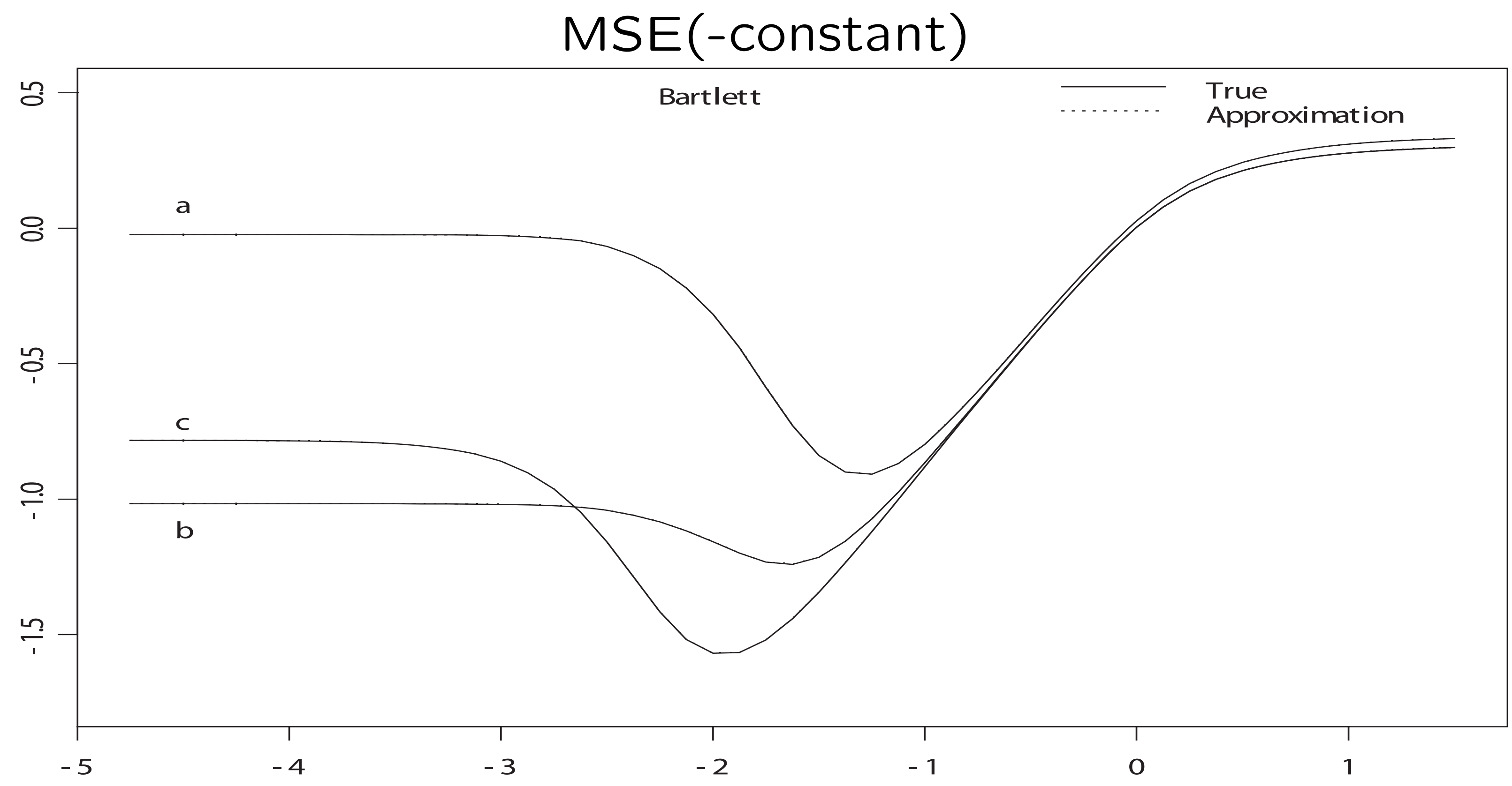
Sample MSE $= (1/500) \sum_{r=1}^{500} \left(estimate^{(r)} - IC^{(r)} \right)^2$



a: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.025)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.05)}$

b: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

c: $\boldsymbol{\lambda_i \Delta = 1/15}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

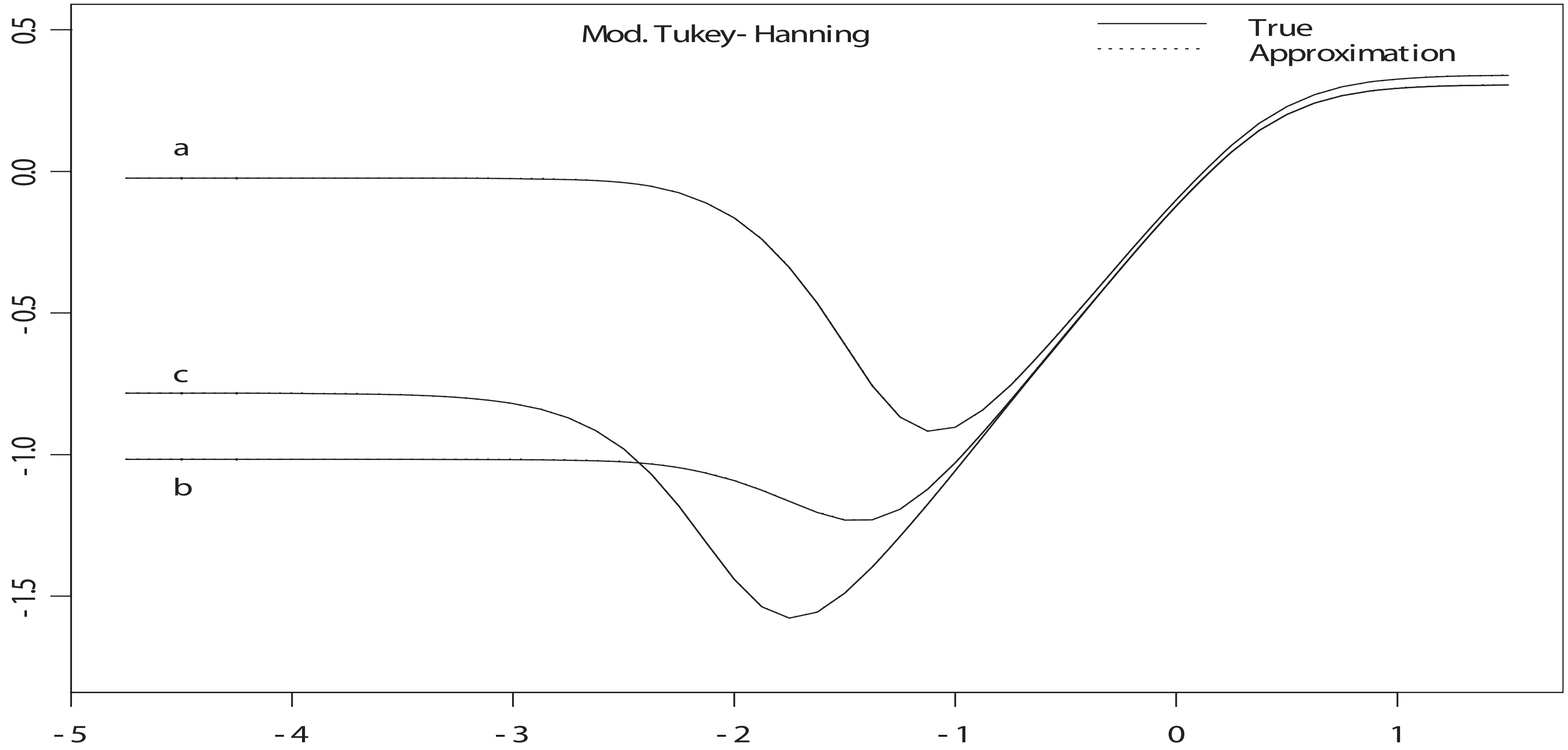


a: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.025)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.05)}$

b: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

c: $\boldsymbol{\lambda_i \Delta = 1/15}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

MSE(-constant)

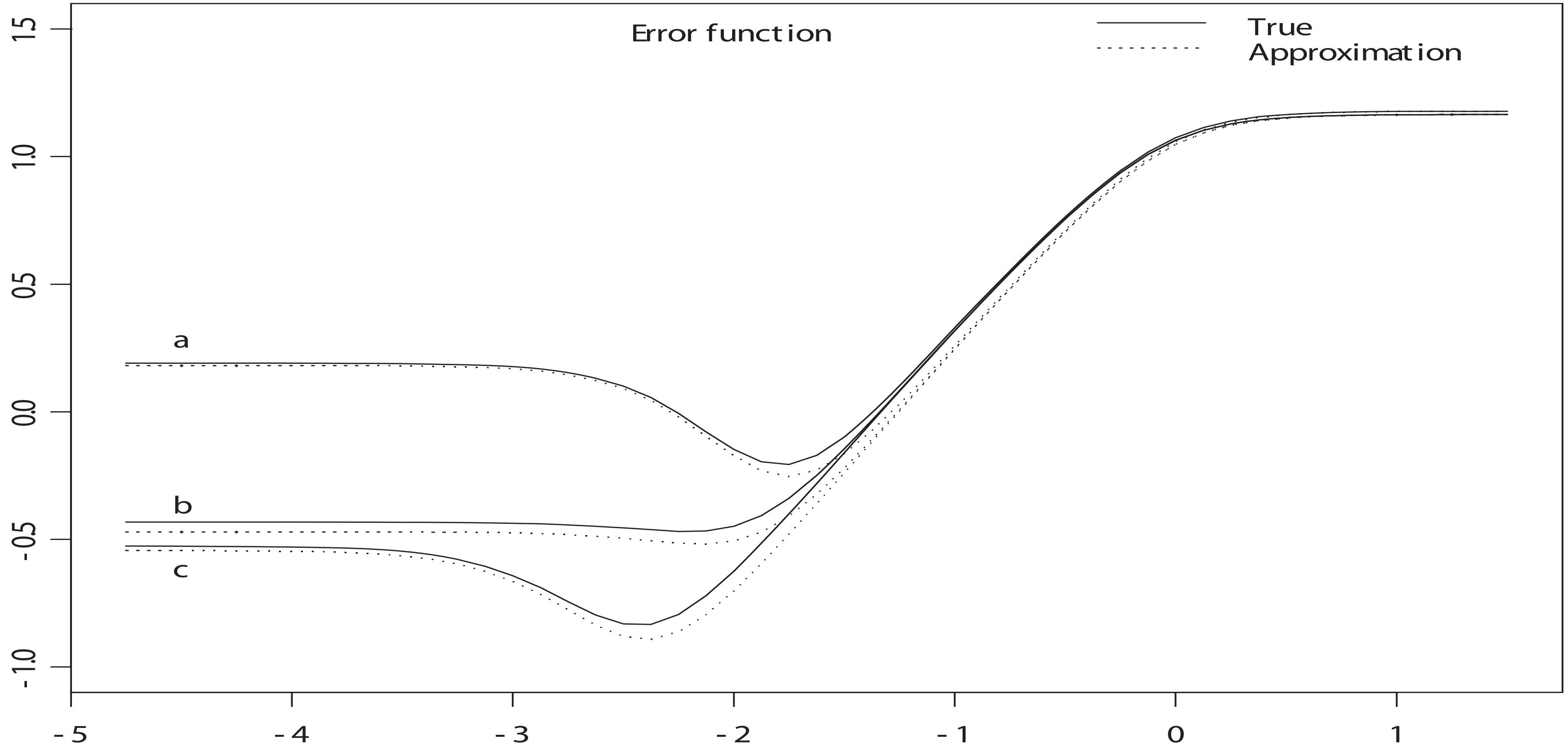


a: $\lambda_i \Delta = 1/60$, $e_1(t_{n_1}) \sim NID(0, 0.025)$, $e_2(t_{n_2}) \sim NID(0, 0.05)$

b: $\lambda_i \Delta = 1/60$, $e_1(t_{n_1}) \sim NID(0, 0.005)$, $e_2(t_{n_2}) \sim NID(0, 0.01)$

c: $\lambda_i \Delta = 1/15$, $e_1(t_{n_1}) \sim NID(0, 0.005)$, $e_2(t_{n_2}) \sim NID(0, 0.01)$

MSE(-constant) $\boldsymbol{\kappa \to 0.1\kappa}$, $\boldsymbol{\gamma \to 10\gamma}$

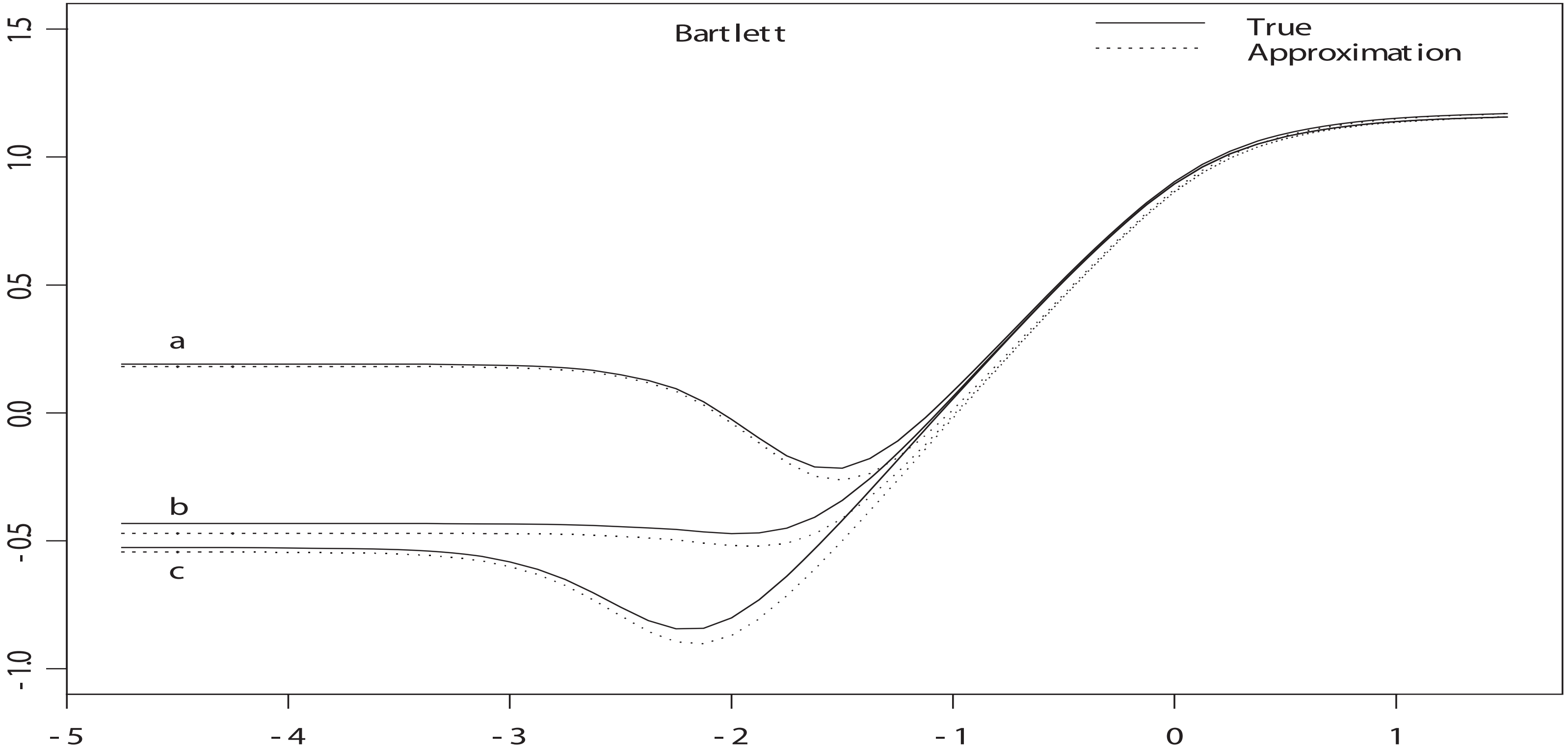


a: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.025)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.05)}$

b: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

c: $\boldsymbol{\lambda_i \Delta = 1/15}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

MSE(-constant) $\boldsymbol{\kappa} \to \mathbf{0.1}\boldsymbol{\kappa}$, $\boldsymbol{\gamma} \to \mathbf{10}\boldsymbol{\gamma}$

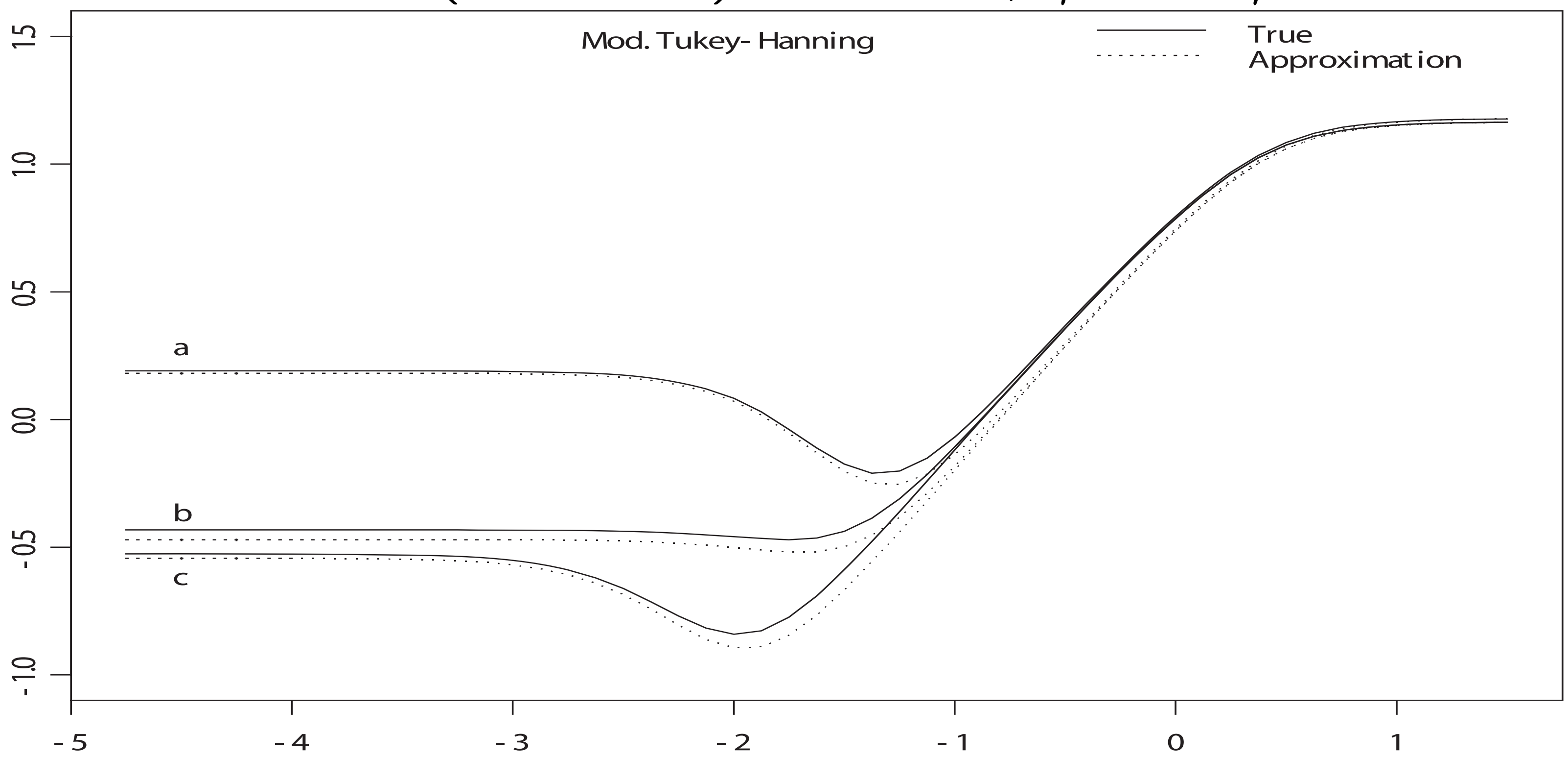


a: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.025)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.05)}$

b: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

c: $\boldsymbol{\lambda_i \Delta = 1/15}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

MSE(-constant) $\boldsymbol{\kappa \to 0.1\kappa}$, $\boldsymbol{\gamma \to 10\gamma}$



a: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.025)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.05)}$
b: $\boldsymbol{\lambda_i \Delta = 1/60}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$
c: $\boldsymbol{\lambda_i \Delta = 1/15}$, $\boldsymbol{e_1(t_{n_1}) \sim NID(0, 0.005)}$, $\boldsymbol{e_2(t_{n_2}) \sim NID(0, 0.01)}$

Summary

Contribution:

■ A framework for evaluating finite sample MSE in the presence of the noise

■ more efficient examples than existing methods (LHY)

Remaining works:

■ More general weight

■ Asymptotic theory