

A Bias Correction Method for Realized Covariance Calculated Using Previous Tick Interpolation

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Introduction

■ Estimation of cross-volatility (Conditional covariance of asset return).

⇒ Portfolio risk, etc.

■ High-frequency data.

e.g. Hourly data, 30 minutes data, \dots , 5 minutes data, Transaction data (Raw data)

⇒ **Realized Covariance**

DGP:

$$\underbrace{dp(t)}_{n \times 1} = \underbrace{\Sigma(t)}_{n \times n} \underbrace{dz(t)}_{n \times 1}, \quad 0 \leq t \leq T$$

(Instantaneous or spot) volatility matrix:

$$\Omega(t) \equiv \Sigma(t) \Sigma(t)'$$

Estimation of integrated volatility $\int_0^T \Omega(t) dt$

⇒ Quadratic variation:

$$\lim_{M \rightarrow \infty} \underbrace{\sum_{m=1}^M \Delta p(mT/M) \Delta p(mT/M)'}_{\text{Realized volatility}} = \int_0^T \Omega(t) dt$$

Intuitively,

$$\int_0^T dp(t) dp(t)' = \int_0^T \Sigma(t) \underbrace{dz(t) dz(t)'}_{dt I_n} \Sigma(t)'$$

Discrete observation

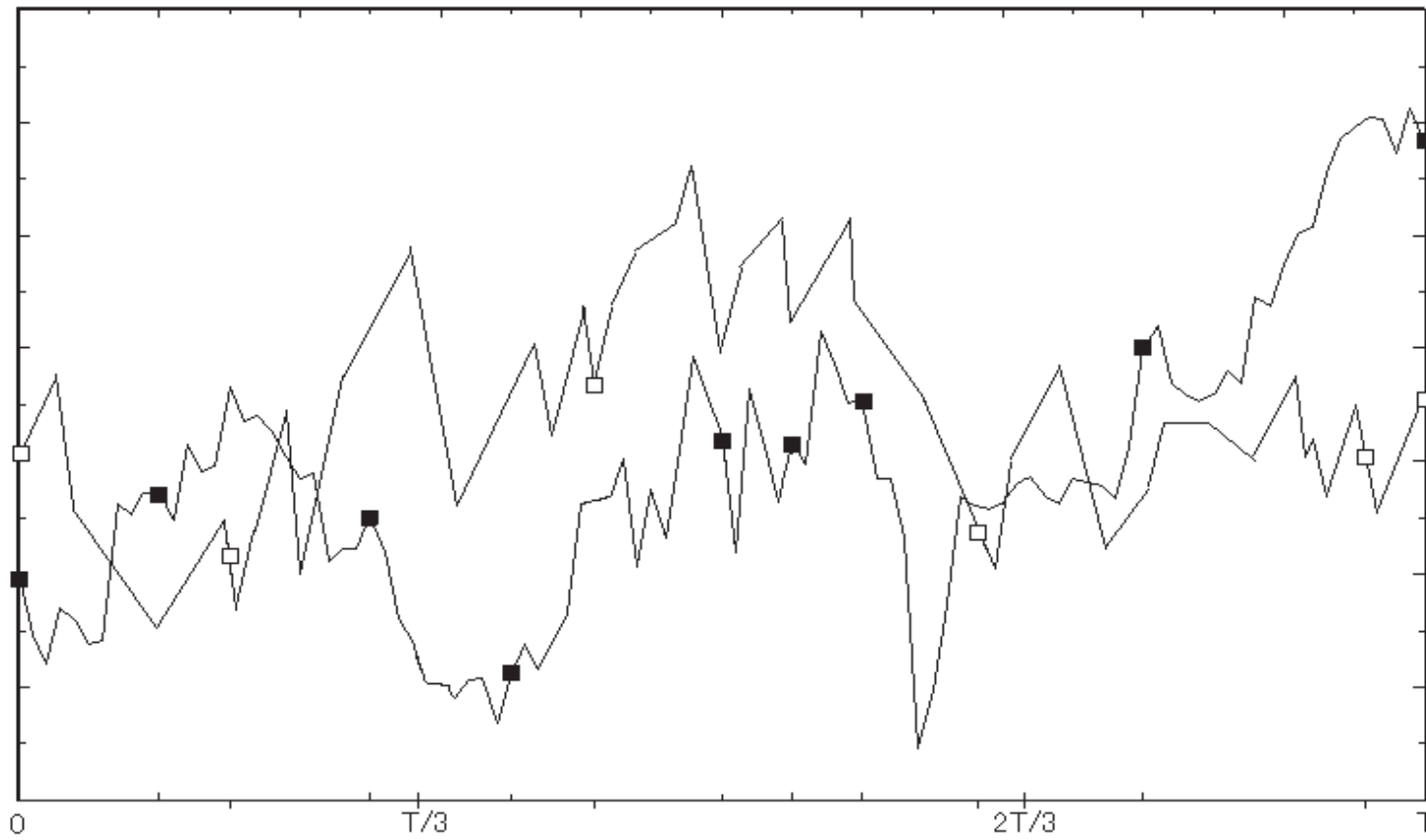
Irregularly observed points:

$$0 = t_0^i < t_1^i < \cdots < t_k^i < \cdots < t_{N_i-1}^i < t_{N_i}^i = T$$

Observed data:

$$p_i(t_0^i), p_i(t_1^i), \cdots, p_i(t_k^i), \cdots, p_i(t_{N_i-1}^i), p_i(t_{N_i}^i)$$

Nonsynchronous observations



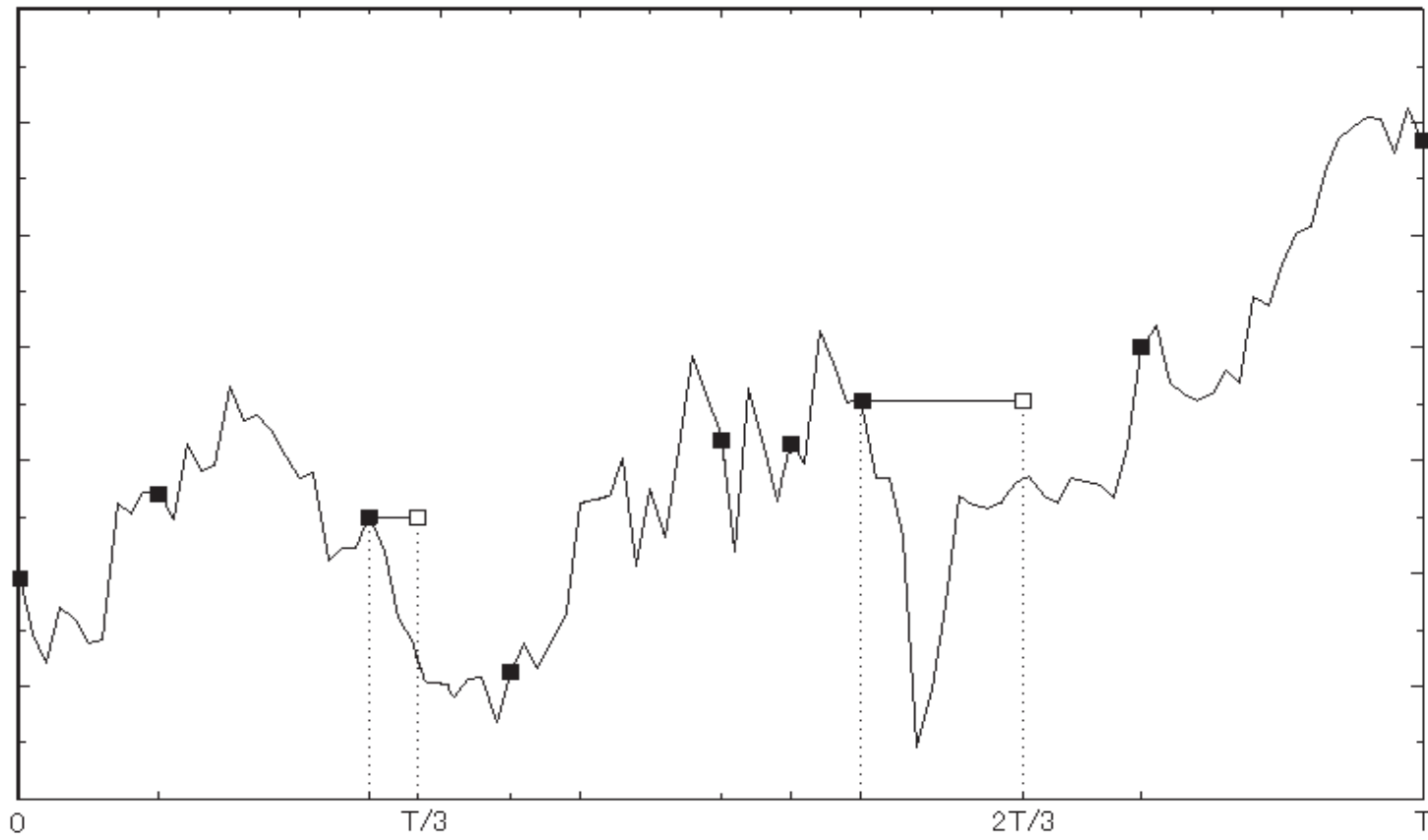
■ 1st asset; □ 2nd asset



Previous-tick interpolation

$$q_i\left(\frac{mT}{M}\right) = p_i\left(\max\left(t_k^i : t_k^i \leq \frac{mT}{M}\right)\right)$$

Previous-tick interpolation



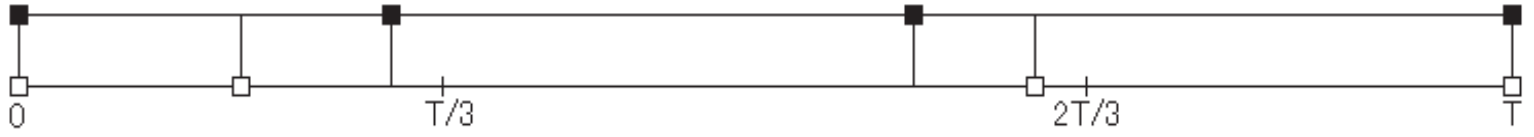
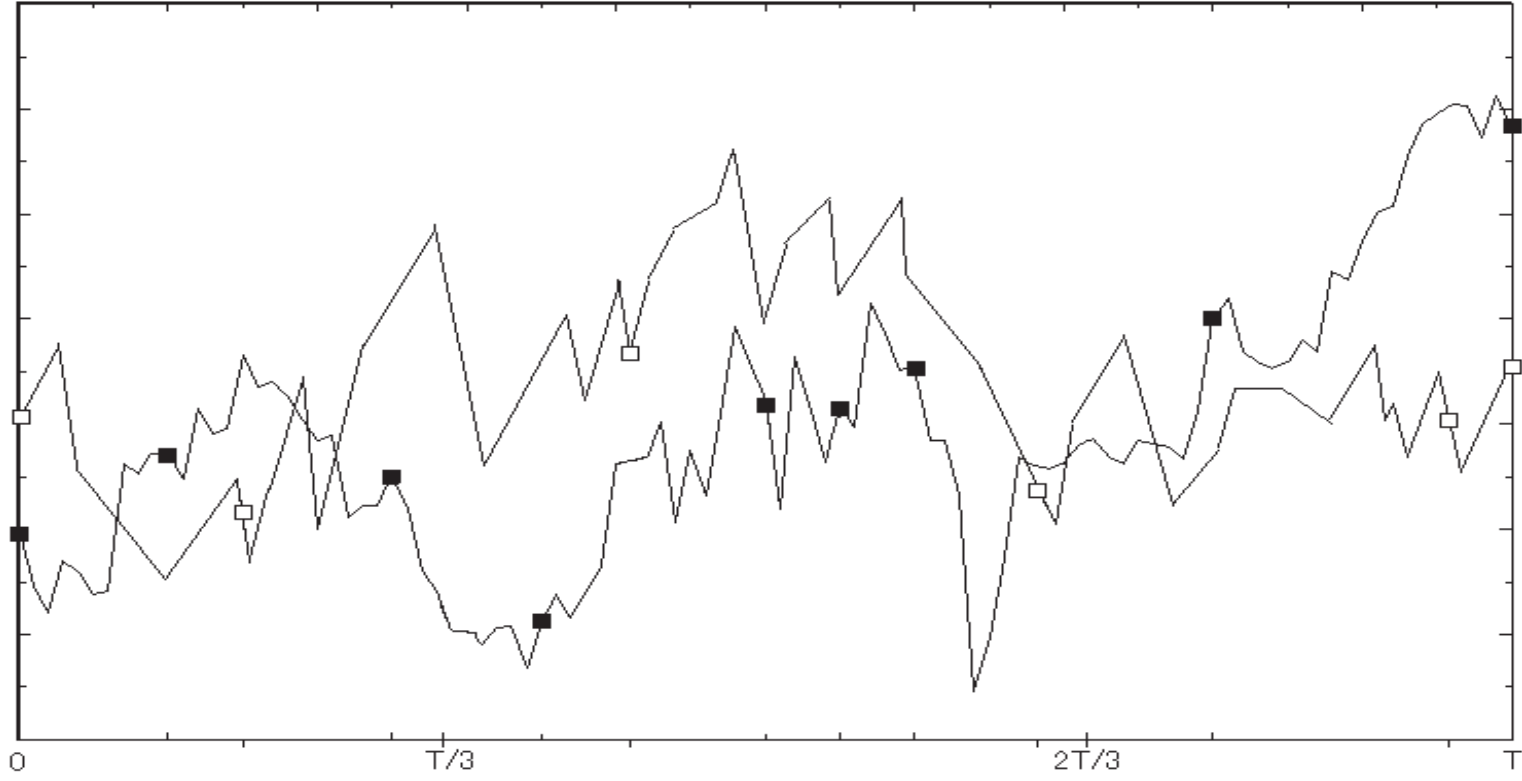
■ Raw data; □ Interpolated data

Realized covariance from the interpolated data

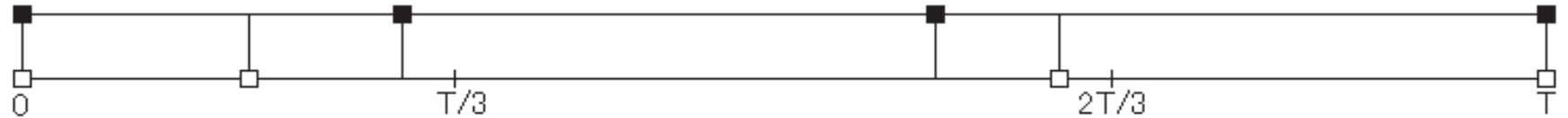
■ Realized covariance

$$\hat{\omega}_{ij} = \sum_{m=1}^M \left\{ q_i \left(\frac{mT}{M} \right) - q_i \left(\frac{(m-1)T}{M} \right) \right\} \left\{ q_j \left(\frac{mT}{M} \right) - q_j \left(\frac{(m-1)T}{M} \right) \right\}$$

⇒ Interpolation bias ⇒ Epps effect (Epps, 1979)



Interpolation bias



$$\begin{aligned}
 & E(\hat{\omega}_{ij}) \\
 = & \sum_{m=1}^M E \left\{ q_i \left(\frac{mT}{M} \right) - q_i \left(\frac{(m-1)T}{M} \right) \right\} \left\{ q_j \left(\frac{mT}{M} \right) - q_j \left(\frac{(m-1)T}{M} \right) \right\} \\
 = & \sum_{m=1}^M \int_{I_m} \omega_{ij}(t) dt
 \end{aligned}$$

Bias-corrected estimator \Rightarrow

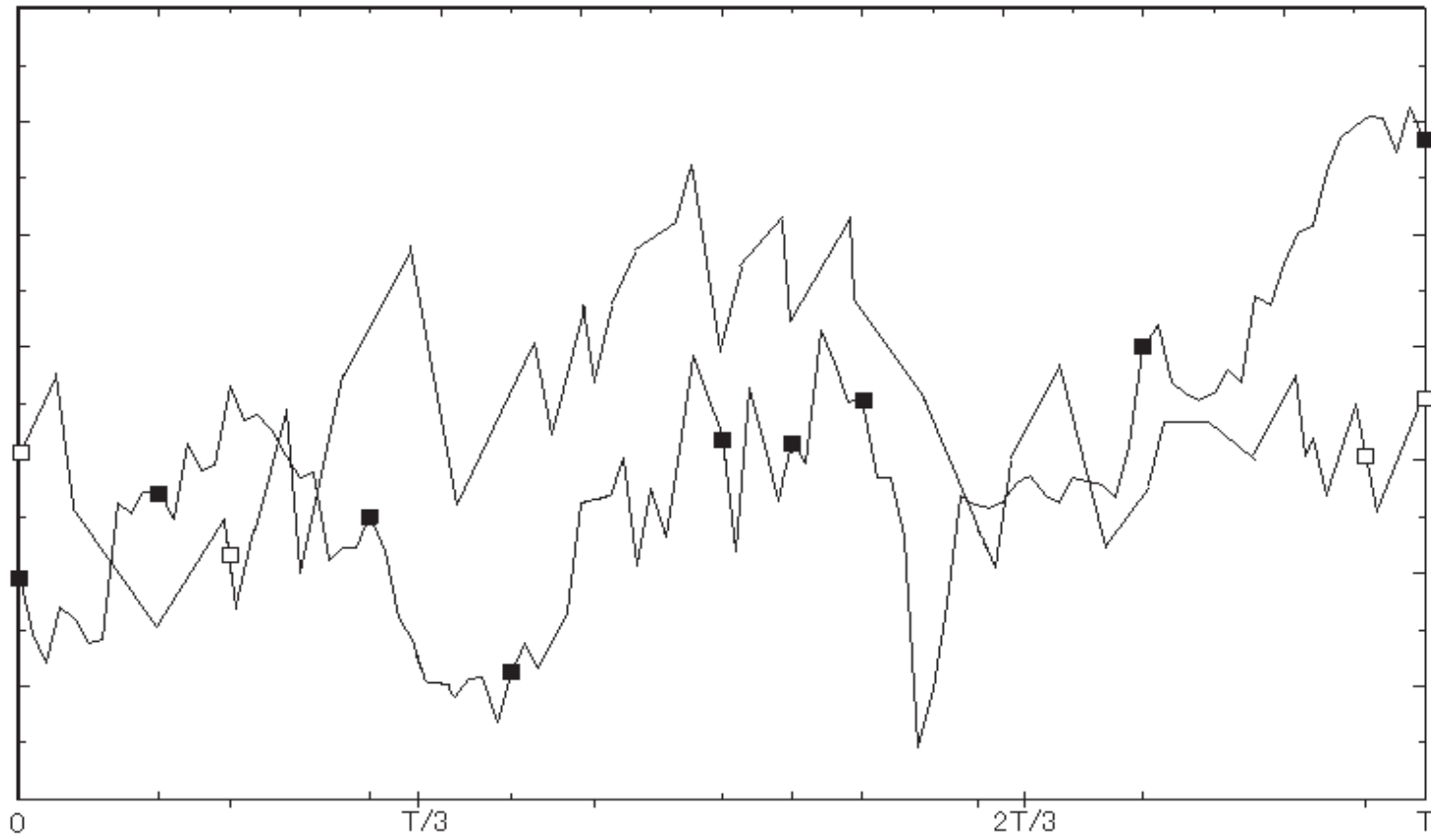
$$\tilde{\omega}_{12} = \hat{\omega}_{12} + \sum_{m=2}^3 \Delta q_1(m-1) \Delta q_2(m) + \sum_{m=2}^3 \Delta q_1(m) \Delta q_2(m-1)$$

Bias-corrected realized covariance

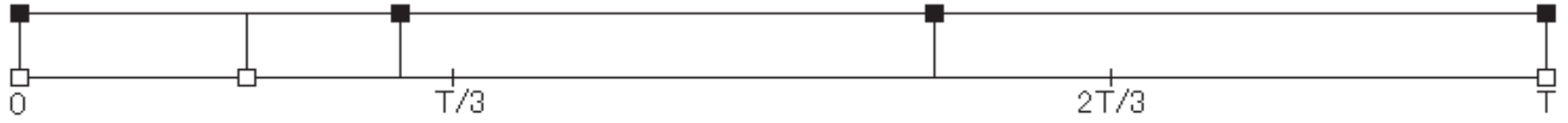
$$\tilde{\omega}_{12} = (\Delta q_1(1) \Delta q_1(2) \Delta q_1(3)) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_2(1) \\ \Delta q_2(2) \\ \Delta q_2(3) \end{pmatrix}$$

$$\tilde{\omega}_{ij} = \Delta q'_i \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \Delta q_j$$

More serious case



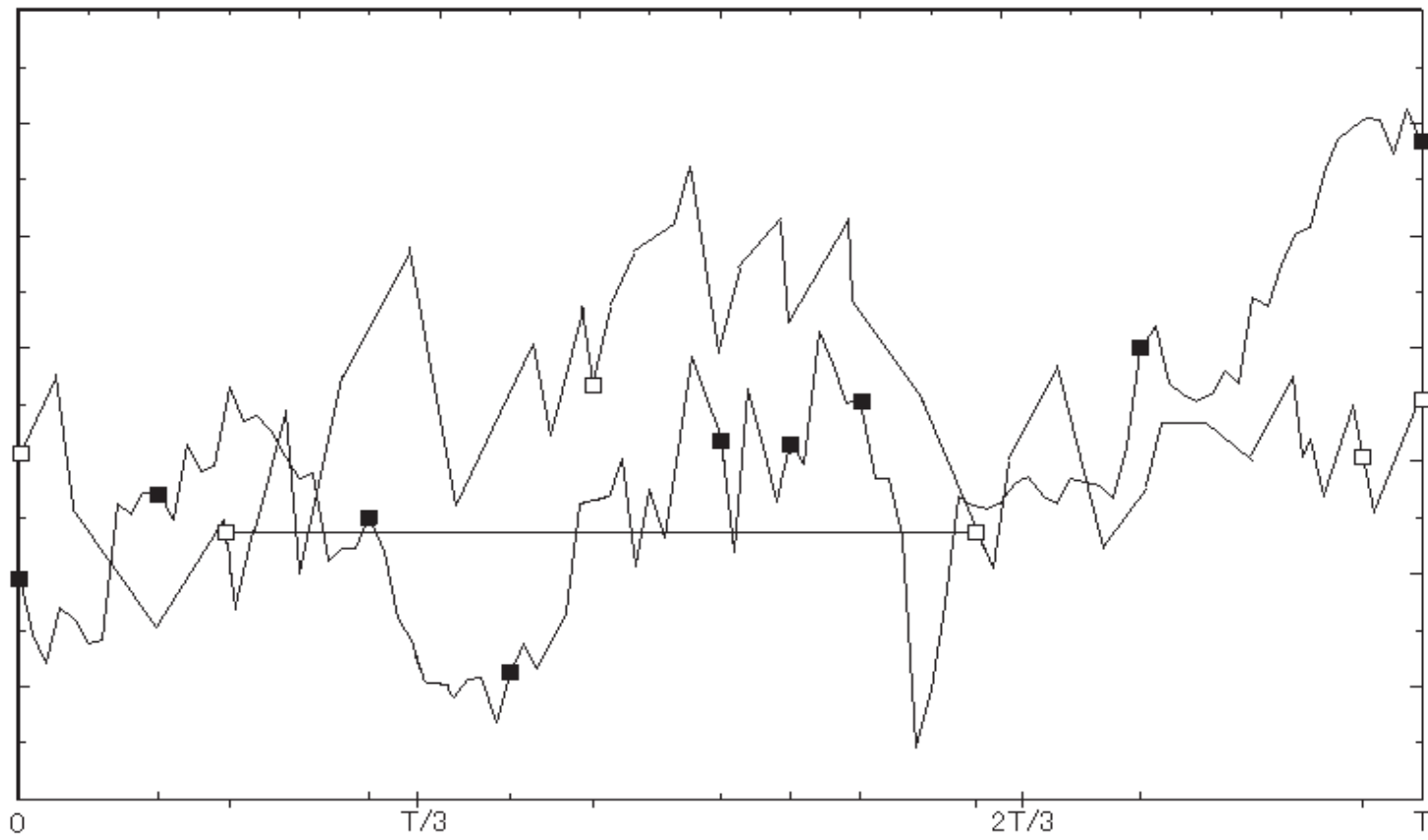
■ 1st asset; □ 2nd asset



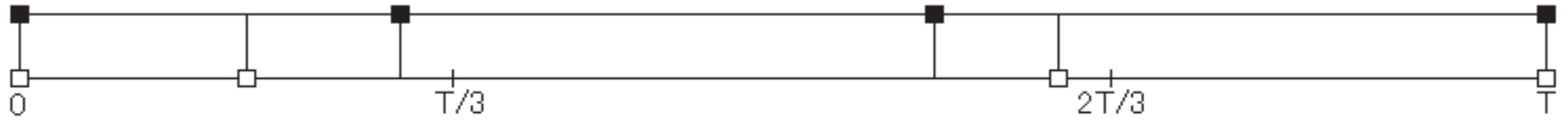
$$\tilde{\omega}_{12} = \hat{\omega}_{12} + \sum_{m=2}^3 \Delta q_1(m-1) \Delta q_2(m) + \sum_{m=2}^3 \Delta q_1(m) \Delta q_2(m-1) + \Delta q_1(1) \Delta q_2(3)$$

$$= (\Delta q_1(1) \Delta q_1(2) \Delta q_1(3)) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_2(1) \\ \Delta q_2(2) \\ \Delta q_2(3) \end{pmatrix}$$

Another serious case



■ 1st asset; □ 2nd asset



$$\tilde{\omega}_{12} = \hat{\omega}_{12} + \sum_{m=2}^3 \Delta q_1(m-1) \Delta q_2(m) + \sum_{m=2}^3 \Delta q_1(m) \Delta q_2(m-1) + \Delta q_1(1) \Delta q_2(3)$$

$$= (\Delta q_1(1) \Delta q_1(2) \Delta q_1(3)) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_2(1) \\ \Delta q_2(2) \\ \Delta q_2(3) \end{pmatrix}$$

$$\begin{aligned}
\tilde{\omega}_{ij} &= \Delta q'_i W \Delta q_j \\
&= \Delta q'_i \begin{pmatrix} 1 & 1 & 1_{1,3} & \cdots & 1_{1,M} \\ 1 & 1 & \cdots & \cdots & \vdots \\ 1_{3,1} & \cdots & \cdots & \cdots & 1_{M-2,M} \\ \vdots & \cdots & \cdots & 1 & 1 \\ 1_{M,1} & \cdots & 1_{M,M-2} & 1 & 1 \end{pmatrix} \Delta q_j
\end{aligned}$$

where

$$W_{kl} = 1_{kl} = \begin{cases} 1 & \text{if } |k - l| \leq 1 \\ 1 & \text{if } k - l > 1 \text{ and } \Delta q_i(k - 1) = \cdots = \Delta q_i(l + 1) \\ 1 & \text{if } l - k > 1 \text{ and } \Delta q_j(l - 1) = \cdots = \Delta q_j(k + 1) \\ 0 & \text{otherwise} \end{cases}$$

Monte Carlo study

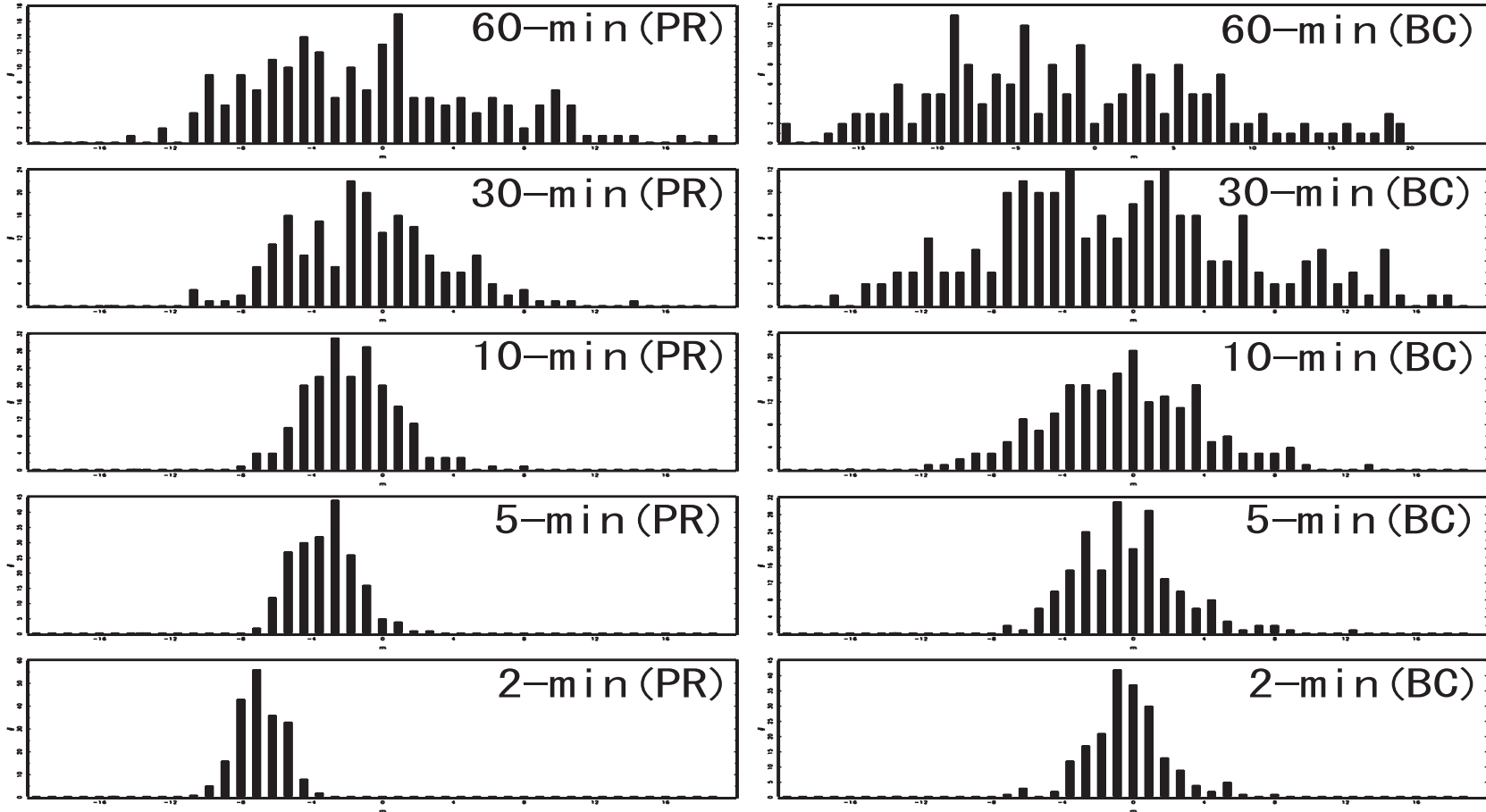
$$\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \quad 0 \leq t \leq T$$
$$d\sigma_{ij}(t) = \kappa (\theta - \sigma_{ij}(t)) dt + \gamma dW_{ij}(t), \quad i, j = 1, 2.$$

where $\kappa = 0.01$, $\theta = 0.01$, and $\gamma = 0.001$ and $T = 60 \times 60 \times 24$ seconds. Time differences are drawn from an exponential distribution with mean 45 seconds for p_1 and 60 seconds for p_2 :

$$F(t_k^i - t_{k-1}^i) = 1 - \exp\{-\lambda_i (t_k^i - t_{k-1}^i)\}, \quad i = 1, 2$$

where $F(\cdot)$ denotes a cumulative distribution function, $\lambda_1 = 1/45$ and $\lambda_2 = 1/60$.

Distribution of measurement error



Summary

Contributions:

- Correction of interpolation bias

Remaining works:

- Evaluation of MSE
- Other microstructure biases

The full paper is available at <http://www.geocities.jp/kanatanit/>