Introduction

The relationship between risk and return is the basic of asset-pricing models. Since Elton (1999), some researchers argued on the use of alternative proxies for the expected return. They believe that average realized return is a poor proxy for the expected return, and it fails to establish the relationship between risk and return. Although the researchers are more concerned about the alternative proxy for expected return (for example, Gebhardt et al. (2001), Brav et al. (2005), Easton and Monahan (2005), Pastor et al. (2008), Easton (2009), Lee et al. (2009), Guay et al. (2011) and Hou et al. (2012)), none of them argued on the reason behind the inability of average realized return as the estimator for the expected return. A number of researchers implicitly assume realized return as a sample of return, and they have conducted empirical analysis of risk and return relationship using the realized return. However, this paper discusses that realized return cannot be the sample of return. If the realized return were not the ex-post realization of the ex-ante expectation, can we use average realized return to estimate the expected return?

For example, Sharpe (1964), Lintner (1965) and Mossin (1966)’s Capital Asset Pricing Model (CAPM) is the first formal derivation of the risk-return relationship. CAPM established that in equilibrium, an asset’s expected return should be related to the associated risk, as measured by beta. To detect the positive risk-return relationship, empirical researchers have been using realized (historical) return data. In text books, the authors presented that the arithmetic mean of the realized returns and the sample variance can be treated as the estimates...
of the \textit{ex-ante} parameters of return, i.e., $\mu$ and $\sigma^2$ respectively, because the realized return is considered as a sample of return.

Following the textbook approach of testing the CAPM, we faced difficulties in explaining the negative average realized return for Tokyo Stock Exchange (TSE). Evidently, the monthly average realized market return for TSE were negative for the samples 1990:1-1994:12 and 2000:1-2003:4. The average monthly realized market return for 1990-94 and 2000-03 were negative at -0.683\% (-8.51\% per year) and -1.71\% (-22.56\% per year), respectively. For these samples, we might wrongly conclude that the market was less risky during the period with negative average realized market return for TSE. Conversely, because of the economic bubble in Japan during 1985-89, the average monthly realized market return for this period was as high as 2.097\% (28.28\% per year). Can this high average realized return be considered as higher risk? Can realized return estimate the expected return, and be able to explain the risk-return relationship? Elton (1999) has argued on the inability of the realized return to explain the risk-return relationship of the market. He noted

‘... in the recent past, the United States has had stock market returns of higher than 30 percent per year while Asian markets have had negative return. Does anyone honestly believe that this was the riskiest period in history for the United States and the safest for Asia?’ (p.1199)

Assuming realized returns as the \textit{ex-post} realization of the \textit{ex-ante} expectation is misleading, and evidently the sample statistics for TSE have failed to estimate the risk-return relationship. Many researchers have considered that realized returns are a sample of returns, i.e., they have assumed realized return as the \textit{ex-post} realization of the \textit{ex-ante} expectation. We believe that \textit{ex-post} value and realized value are different.

The difference between the realized value and the \textit{ex-post} value is resulted from the presence of the information sets of different periods. For the (\textit{ex-ante}) return at $t$, investors
are considering \((t+1)\)'s information sets available at \(t\). As the price at \(t\) and \((t+1)\) are derived from the information sets of \((t+1)\) and \((t+2)\) respectively, realized return at \((t+1)\) incorporates the information set from both \((t+1)\) and \((t+2)\). In contrast, information set on \((t+2)\) is not incorporated in the \textit{ex-post} return at \((t+1)\). Rather, the \textit{ex-post} return at \((t+1)\) incorporates the information of \(t\) to \((t+1)\). Realized return cannot be the \textit{ex-post} realization of the \textit{ex-ante} expectation because of the differences of the information sets in the \textit{ex-post} return and the realized return at \((t+1)\).

Returning back to the origin of the asset-pricing model, in this paper, we redefine the \textit{ex-ante} value, \textit{ex-post} value and the realized value.

As the price is the discounted value of next period's expected price, we have shown that realized return cannot be a sample of return. We argue that considering \textit{realized return as the sample of return is a fallacy} that has misled the empirical researchers in estimating the expected return and risk. We show that the price cannot be the \textit{ex-post} realization of the \textit{ex-ante} expectation. The objective of this paper is to identify the reason behind the inability of the realized returns as a sample of returns.

Elton (1999) has concluded that because of the presence of information surprises, arithmetic mean of the realized returns is not a good proxy for the expected return. Like Elton (1999), Gebhardt et al. (2001), Brav et al. (2005) and others, we also believe that the average realized return fails to estimate the expected return. Unlike Elton (1999), we have shown that the information sets in realized return and the \textit{ex-post} return are different. Our paper is the first to explain the failure of the use of realized returns from the pricing point of view.

In section 2 we show that realized return cannot be a sample of return. We show the difference between the realized return and the \textit{ex-post} return because of the different information sets. We also provide an alternative definition of the \textit{ex-post} return in this section. In section 3, we present graphical explanation from the pricing point of view and showed the validity of CAPM even when the average realized return fails to estimate the expected return. Section 4 follows the conclusion.

### II The \textit{ex-ante} return, \textit{ex-post} return and the realized returns:

The main focus of the asset-pricing model is to explain the risk-return relationship. Theoretically, we can establish risk-return relationship (for example, CAPM). However, unobservable nature of the \textit{ex-ante} expected return hinders estimating the empirical risk-return relationship. As a result, in empirical analysis, most of the researchers consider realized return as the \textit{ex-post} realization of the \textit{(ex-ante)} return, i.e., they assume realized return as a sample of return. For example, they assume that \((\text{ex-ante})\) returns\(^1\) are normally distributed with mean \(\mu_i\) and variance of \(\sigma_i^2\). They have used the average realized return and sample variance as estimators of the \textit{ex-ante} expected return and the \textit{ex-ante} variance. Nevertheless results of the empirical analysis were almost inconclusive.

Some researchers intuitively believe that the realized return cannot be the \textit{ex-post} realization of the \textit{(ex-ante)} return and consequently empirical estimation differs from the \textit{ex-ante} expectation. In this section we depict the in-

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\(^1\) In general, expected return has been considered as the \textit{‘ex-ante return’} by the researchers. As we have discussed later in the paper that the \textit{ex-ante} literally means the random future values. If we define \textit{ex-ante} return as the expected return, we are disregarding the randomness of the future values. Therefore, we have defined the returns as \textit{‘(ex-ante) returns’} in this paper instead of the \textit{‘expected returns’} as has been considered by the other researchers.
ability of the realized return as the \textit{ex-post} realization of the \textit{ex-ante} and present that \textit{ex-post} value is different from the realized value. We portray our argument from the pricing point of view and in doing so we show that the information set in the price is different from the information set in the \textit{ex-post} value. Our argument is based on the following simplified assumptions:

(i). In an one-period setting, price is the discounted value of the next period’s expected price, \( p_i = \frac{E(p_{i+1})}{d_i} \) where \( d_i > 1 \). In addition, we assume that \( E(\widehat{p}_{i,t+1}) \) incorporates all future information available at \( t \).

(ii). The state of future economy changes with time.

Assumption (i) states that for any risky asset the investors are assumed to expect positive payoffs in future and can be considered as one of the basic assumptions in valuation. Assumption (ii) can be considered as the base of our argument. Most of the researchers assume a steady state of the economy where there is no change in the fundamental economic variables. Rather they consider any change in the information set (surprises) as a change in variables other than the fundamentals. And for a sample, these surprises are expected to be cancelled out. We assume that any change in the economy is a result of the changes in the economic variables, both fundamentals as well as firm specific ones. This may lead us to assume that investors’ forecasts about the asset’s expected price would increase (decrease) with forecasted positive (negative) changes in the economic variables. Besides, researchers have been using the realized return in empirical tests, and in reality the economy is changing also. Thus our second assumption is much closer to the reality.

Most of the researchers have been using realized return in establishing the empirical risk-return relationship; we introduce 2 scenarios and argue on the inability of the realized return to explain the risk-return relationship. As we proceed, we discuss the different information sets in the asset-pricing, and gradually, we present the difference between the realized value and the \textit{ex-post} value. We conclude that realized return cannot be a sample of return.

\section*{2-1: An example:}

We begin with a simple example for better understanding of our argument. We show that when the assumptions (i) and (ii) hold, average realized returns cannot estimate the expected return. Investor’s expected price would rise (fall) with the favorable (unfavorable) future economic forecasts. We start our argument with a series of unfavorable future economy in scenario 1. Under one-period model settings, we assume that price in every period is formed based on the expected price of the next period. Let us assume the expected prices for \((t+1)\) to \((t+4)\) at \( t \), \((t+1)\), \((t+2)\) and \((t+3)\) are 105, 95, 89 and 83 respectively. If we assume 5\% expected return\(^2\) for the investors, we would get the price for \( t \) to \((t+3)\) as \((105/1.05)\), \((95/1.05)\), \((89/1.05)\) and \((83/1.05)\) respectively. For this series the average realized return would be negative. Note that our expected return is 5\% in scenario 1. The sample average realized return for these types of series cannot estimate the expected return of 5\%. Why does average realized return fails to estimate the expected return?

\begin{itemize}
\item \(2\) Although the expected rate of return (the discount rate) might change with the changes in the economic forecasts, for simplicity, we consider constant discount rate in this paper. Note that, the argument of this paper can support a model with changing discount rate scenario also.
\end{itemize}
Scenario 1: Realized return and the risk-return relationship in downward Market

This table forecasts the future values from \((t+1)\) to \((t+4)\) in a down-ward market. The expected return (cost of capital) is 5\% (i.e., discount rate, \(d=1.05\)). For simplicity of the argument we assume expected return as constant.

<table>
<thead>
<tr>
<th></th>
<th>(t)</th>
<th>(t+1)</th>
<th>(t+2)</th>
<th>(t+3)</th>
<th>(t+4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E(\hat{p}_{t+\tau}))</td>
<td></td>
<td>105</td>
<td>95</td>
<td>89</td>
<td>83</td>
</tr>
<tr>
<td>(p_{t+\tau})</td>
<td>105 / 1.05</td>
<td>95 / 1.05</td>
<td>89 / 1.05</td>
<td>83 / 1.05</td>
<td>…</td>
</tr>
<tr>
<td>(r_{t+\tau+1} = \frac{p_{t+\tau+1}}{p_{t+\tau}})</td>
<td>0.905</td>
<td>0.937</td>
<td>0.933</td>
<td>…</td>
<td></td>
</tr>
</tbody>
</table>

Scenario 2: Realized return and the risk-return relationship in upward Market

This table forecasts the future values from \((t+1)\) to \((t+4)\) in an up-ward market. The expected return (cost of capital) is 5\% (i.e., discount rate, \(d=1.05\)). For simplicity of the argument we assume expected return as constant.

<table>
<thead>
<tr>
<th></th>
<th>(t)</th>
<th>(t+1)</th>
<th>(t+2)</th>
<th>(t+3)</th>
<th>(t+4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E(\hat{p}_{t+\tau}))</td>
<td></td>
<td>105</td>
<td>120</td>
<td>135</td>
<td>150</td>
</tr>
<tr>
<td>(p_{t+\tau})</td>
<td>105 / 1.05</td>
<td>120 / 1.05</td>
<td>135 / 1.05</td>
<td>150 / 1.05</td>
<td>…</td>
</tr>
<tr>
<td>(r_{t+\tau+1} = \frac{p_{t+\tau+1}}{p_{t+\tau}})</td>
<td>1.143</td>
<td>1.125</td>
<td>1.111</td>
<td>…</td>
<td></td>
</tr>
</tbody>
</table>

For \((t+1)\) in scenario 1, researchers would consider \((95/1.05)\) as the ex-post realization of ex-ante price for \(t\), i.e., \((95/1.05)\) is treated as a realized value of the ex-ante distribution of future price of \(\hat{p}_{t+1}\) at \((t+1)\) for \(t\). Can \((95/1.05)\) at \((t+1)\) be a ex-post value of the future price of \((t+2)\) for \(t\)?

The price at \((t+1)\) is the discounted expected price of \((t+2)\). In this example, the price \((95/1.05)\) at \((t+1)\) is derived from the information on the future price for \((t+2)\) which is available at \((t+1)\). In general, the ex-post value at \((t+1)\) is the observed value from the information on \(t\) to \((t+1)\). \((95/1.05)\) cannot be the ex-post value at \((t+1)\) as this value is derived from the information of \((t+2)\) instead of the information set of \(t\) to \((t+1)\). Under assumption (ii), the expected price of \((t+2)\), \(E(\hat{p}_{t+2})\), has no relation to the distribution of \(\hat{p}_{t+1}\) at \((t+1)\). So the realized return can neither be the
ex-post return nor the sample of return.

In scenario 2, with favorable economic forecasts, the expected values increase from 105 in \( t+1 \) to 150 in \( t+4 \). We can consider scenario 2 as an illustration of the Japanese bubble during 1985-90. With this increase, the prices also increase from 100 at \( t \) to 143 in \( t+3 \). The average realized return for this type of upward series will be much greater than the expected return of the asset (5% in this case). Besides, as we have argued before, \((120/1.05)\) cannot be considered as the ex-post value at \( t+1 \) because \((120/1.05)\) is derived from the information set on the expected price of \( t+2 \) available at \( t+1 \).

None of the researchers have argued on the information sets in the price as well as in the ex-post return. In this section, with simple illustrative examples under assumptions (i) and (ii), we have shown that the information sets in price and in ex-post return are different, and price cannot be considered as the ex-post realization of the ex-ante expectation.

2-2 : Realized return and the ex-post return:

In this section, we provide a general discussion on the difference between realized return and ex-post return. We have divided information at \( t \) into two parts for better understanding, and we define information as:

\[
\Phi_t = \Phi^H_t + \Phi^{F+1}_t
\]

where, \( \Phi_t \) is the total information set available at \( t \), \( \Phi^H_t \) is the past information set on \( t-1 \) to \( t \) available at \( t \), and \( \Phi^{F+1}_t \) is the future information set on \( t+1 \) that is incorporated at \( t \). Past information set is assumed to be comprised of the results of the operating activities between \( t-1 \) to \( t \). In contrast, the economic information as well as the firm’s future policies is incorporated in the future information set.

Under assumption (i), price \( p_{i,t} \) is the discounted value of \( E(\tilde{p}_{i,t+1}|\Phi^{F+1}_t) \). Similarly, price \( p_{i,t+1} \) is the discounted value of \( E(\tilde{p}_{i,t+2}|\Phi^{F+2}_{t+1}) \).

At \( t+1 \), \( p_{i,t+1} \) does not incorporate past information set \( \Phi^{F+2}_t \) it is derived from the future information set of \( \Phi^{F+2}_{t+1} \). The following figure explains the difference between information sets in price and the ex-post value.

In the following discussion, we provide further explanation to confirm that the realized return cannot be the ex-post return. We define \( \text{(ex-ante)} \) return at time \( t \), \( \tilde{r}_{i,t+1} \) as,

\[
\tilde{r}_{i,t+1} = \frac{\tilde{p}_{i,t+1}|\Phi^{F+1}_t}{\tilde{p}_{i,t}|\Phi_{t}}
\]

and, the realized return, \( r_{i,t+1} \), is defined as,

\[
r_{i,t+1} = \frac{p_{i,t+1}|\Phi^{F+2}_{t+1}}{p_{i,t}|\Phi_{t}}
\]

The researchers consider \( (\tilde{p}_{i,t+1}|\Phi^{F+2}_{t+1}) \) as the ex-post realization of \( (\tilde{p}_{i,t+1}|\Phi^{F+1}_t) \), i.e. they have been assuming \( (\tilde{p}_{i,t+1}|\Phi^{F+2}_{t+1}) \) as the sam-

\[3\] Past information, for example as cited by Elton (1999), high earnings announcements of MacDonald, has little or no role in forming future expectation of the investors. Does high earnings announcement really lead to higher future price?

In TSE, the annual earnings for Nintendo was the highest in March of 2009 at JPY 279 billion (approx); the price dropped from JPY 71,900 in 2007:10 to JPY 23,180 in 2009:10 following the earnings information, however.

If positive (negative) past information has an impact on the following price, the price would have increased (decreased) following the information. The drift in Nintendo’s price, even with the highest earnings information, can be an example of the absence of the effect of past information on the price.
ple of the distribution of future random price of \( \tilde{p}_{i,t+1} | \Phi^{Fr+1} \). The return at \( t \) in equation (i) incorporates the information about the time period \((t+1)\), available at \( t \). In equation (2), \( (p_{i,t+1} | \Phi^{Fr+2} ) \) has no relation with \( (\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \) under assumption (ii), however. Instead, \( (p_{i,t+1} | \Phi^{Fr+2} ) \) is the discounted expected value of \( (p_{i,t+2} | \Phi^{Fr+2} ) \). The realized return of \((t+1)\) in equation (2), includes information about periods \((t+1)\) and \((t+2)\) for \( (p_{i,t} | \Phi^{Fr+1} ) \) and \( (\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \) respectively. The information sets in \( \tilde{r}_{i,t+1} \) and \( r_{i,t+1} \) are different. These two values are derived from different information sets of different time periods. As a result realized return can neither be \textit{ex-post} return nor a sample of return.

2-3 : Alternative Definition of \textit{ex-post} Value:

In section 2.2, we have shown that the present belief about the \textit{ex-post} return is misleading. How can we measure the \textit{ex-post} return? At \( t \), we consider, the \textit{ex-ante} prediction follows,

\[
\tilde{p}_{i,t+1} = (p_{i,t} | \Phi^{Fr+1} ) + (\tilde{x}_{i,t+1} | \Phi^{Fr+1} ) \tag{3}
\]

where, \( (\tilde{x}_{i,t+1} | \Phi^{Fr+1} ) \) is random operating value for \( t \) to \((t+1)\) based on available information set \( \Phi^{Fr+1} \) at \( t \). We assume earnings, \( \tilde{x}_{i,t+1} \), as the random operational outcome from \( t \) to \((t+1)\) realized at \((t+1)\). We observe earnings for \( t \) to \((t+1)\), i.e.; \( (\tilde{x}_{i,t+1} ) | \Phi^{Hr+1} \). Thus, we define the \textit{ex-post} value at \((t+1)\), \( (p^{*}_{i,t+1} | \Phi_{Hr+1} ) \) as,

\[
p^{*}_{i,t+1} = (p_{i,t} | \Phi^{Fr+1} ) + (\tilde{x}_{i,t+1} | \Phi^{Hr+1} ) \tag{4}
\]

where, \( \tilde{x}_{i,t+1} \) is the observed earnings at \((t+1)\). The value in equation (4) is the realized value of \textit{ex-ante} random price of \( (\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \) for \((t+1)\) made at \( t \). The realized price, \( p^{*}_{i,t+1} \), is not the \textit{ex-post} realization of \( (\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \); where-as, \( x^*_{i,t+1} \) is the observed earnings for \( t \) to \((t+1)\) at \((t+1)\). The \textit{ex-post} return, \( r^{*}_{i,t+1} \), can be written as:

\[
r^{*}_{i,t+1} = \frac{p^{*}_{i,t+1} | \Phi_{Hr+1} }{(p_{i,t} | \Phi^{Fr+1} )} \tag{5}
\]

As a concluding remark of section 2, the \textit{ex-ante} value at \( t \) is the expected value of \( E(\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \) for \((t+1)\). \( E(\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \) is discounted to derive \( p_{i,t} \) at \( t \). The \textit{ex-post} value at \((t+1)\) is the realized (observed) value of time \( t \)'s anticipation of \( (\tilde{p}_{i,t+1} | \Phi^{Fr+1} ) \) that we would observe as we move to \((t+1)\). In contrast, price \( p_{i,t+1} \) is derived from \( E(\tilde{p}_{i,t+1} | \Phi^{Fr+2} ) \) at \((t+1)\). In section 2.2 we have argued that the information sets in these values are different. At \((t+1)\), price \( p_{i,t+1} \) incorporates the information set \( \Phi^{Fr+2} \) on \((t+2)\), whereas the \textit{ex-post} value at \((t+1)\), \( p^{*}_{i,t+1} | \Phi_{Hr+1} \), is observed from the operational activities of \( t \) for \((t+1)\).

III The Realized Return and CAPM:

When realized return is assumed as the \textit{ex-post} realization of the \textit{ex-ante} return, the estimate of the expected return becomes negative whenever the sample average realized return is negative. For TSE, average market return during 1990-94 and 2000-03 were negative. The researchers have considered this period as the adjustment period after the economic bubble during late 80s. They have argued that rational pricing mechanism was absent during this period, and any test on this sample would be inconclusive. As a result, researchers have discarded these periods from the empirical tests of asset-pricing and they have explored alternative models. However, under
assumptions (i) and (ii), from the pricing point of view we can present that CAPM may be valid and can explain the risk-return relationship also. In figure 3, we present the scenario of pricing in a downward market.

For this series, the realized return is negative
The price is derived by discounting the next period's expected value. The expected return for the risky asset should be positive. But for a down-ward market, the realized return becomes negative because of the fall in the expectation of the future price. Thus any inference based on the realized return will empirically invalidate CAPM, whereas the model is valid from the pricing point of view.

On the other hand, during economic bubble an asset’s expected value increases with the forecasted economic information. As a result, the realized return becomes much higher than the expected return for the investors in figure 4. During economic bubble, if we use the use of average realized return to estimate the expected return we would get much higher estimate for the expected return. As a result the empirical result will be inconclusive.

Under assumptions (i) and (ii), in both scenarios, figure 3 and figure 4 support the validity of CAPM as the expected return is positive. If we use realized returns, we would get higher (negative) estimate for the expected return during bubble (downward market). , the empirical tests under both of these cases, the use of realized return might invalidate CAPM. We have shown in this paper that realized return cannot be used as a sample of return to estimate the expected return.

IV Conclusion:

In this paper, we focused on the belief of considering realized return as a sample of return. Under assumptions (i) and (ii), we have shown that realized return cannot be the ex-post realization of the ex-ante expectation.

The researchers can establish the risk-return relationship in theory. The unobservable nature of the expected return has led the empirical researchers to use realized return as a sample of return. And the measurement of the empirical risk-return relationship has been inconclusive and controversial. As a result, a number of researchers have introduced new models to measure the empirical risk-return relationship.

For example, Fama and French (1992) have introduced the 3-factor model in an attempt to explain the empirical risk-return relationship. Their model gained popularity as they focused on forming an empirical model that would fit the realized return data. The model is used to explain the ex-ante risk-return relationship from the realized return data. We have shown that realized return can neither be the ex-post return nor the sample of return. What economic implication does the realized return data contain?

Our paper is the first one to explicitly define the ex-post value, and we have shown that realized value and the ex-post value are different because of the differences in the information sets. We conclude that realized return cannot be the ex-post realization of the (ex-ante) return, i.e., realized return cannot be a sample of return.

References


4) During a recession, the increase in risk expectation will increase the expected return of an asset resulting in a fall in equity price of the asset (a value loss). And as the inference based on ex-post returns depends on the properties of the particular data under examination, by averaging ex-post realized returns over the course of a recession, one might wrongly conclude that the asset is less risky because of its lower ‘expected’ returns (Campello et al. (2008)).


If realized return is not the ex-post realization of the ex-ante expectation, can we use average realized return to estimate the expected return? In textbooks, the authors treat realized return as a sample of return. In this paper, we redefine realized return and the ex-post return, and we argue that realized return cannot be the sample of return. This paper is the first to explain the difference between ex-post and realized return; and we show the inability of the realized return to be the ex-post realization of the ex-ante expectation of return.

In this paper we go back to the basics of the asset-pricing model, for example, and we focus on the reason behind the failure of realized return as an estimator for expected return. We show that, in general, realized return cannot be a sample of return and using the average realized return as the estimator for the expected return has been misleading over the years.

Keywords: Capital Asset Pricing Model; realized return; ex-ante return; ex-post return.

JEL Classification: G12